

Paper - III

M.Sc  
Part - III

**NUMERICAL**

**MATHEMATICS**

**ANALYSIS**

# University of Gujrat

New Campus, Hafiz Hayat, Gujrat

|                     |   |
|---------------------|---|
| Course Title :      | Numerical Analysis-I  |
| Course Code :       | MATH-401  |
| Instructor :        | Dr. Jamshad Ahmad   |
| Recommended Books : | <ol style="list-style-type: none"><li>1. Numerical Analysis, Richard L. Burden, J. Douglas Faires, (9th Edition), 2010, Brooks/Cole Publishing Company.</li><li>2. Numerical Methods, S.R.K. Iyenger R.K. Jain, New Age International Publishers.</li><li>3. Dr.Saeed Akhtar Bhatti, Numerical analysis with C++, Edition Lates,t Publisher Urdu Bazar lahore</li></ol>   |
| Reference Books:    | <ol style="list-style-type: none"><li>4. Applied Numerical Analysis, Curtis F. Gerald, Patrick O, Wheatley, (7th Edition), 2003, Addison Education.</li><li>5. Numerical Methods for Mathematics, Science and Engineering, John H. Mathews, (4th Edition), 2004, Prentice Hall International.</li><li>6. Numerical Methods, V. N. Vedamurthy, Ch. S. N. Iyenger, 2002, Vikas Publishing House PVT Ltd.</li></ol>  |
| Email ID :          | jamshad.ahmad@uog.edu.pk  |
| Lectures            | 32 sessions of 90 minutes each  |
| Attendance Policy   | <p><b>A minimum of 70% attendance is required for a student to be eligible to take the final examination.</b></p> <p>The students with less than 70% of the attendance in a course shall be given the grade SA (Short Attendance) in such a course and shall not be allowed to take its End Term Exams and will have to reappear in the course to get the required attendance to be eligible to sit in the exam when the course is offered the next time.</p> |
| Grading             | <p>The course will be evaluated on the basis of the following percentage:</p> <ul style="list-style-type: none"><li>• Mid Term 25%</li><li>• Sessional work 25%<ul style="list-style-type: none"><li>o Presentation/Practical 5%</li></ul></li></ul>  |

# 16 WEEK PLAN

|    |  |
|----|--|
| 1  | Introduction to Numerical Analysis, Number System and errors; Round off Errors and Computer Arithmetic. Error estimation.                                |
| 2  | Floating point Arithmetic, Algorithm and Convergence Solution of non Linear Equations: Iterative Methods and Convergence.                                |
| 3  | Bisection Methods, Fixed point iterative Method, <b>QUIZ 1</b>   |
| 4  | Regular Falsi, Secant and Newton's Method, + <b>Assignment#1</b>   |
| 5  | Gaussian Elimination methods, Gauss-Jordan Method, <b>QUIZ 2</b>   |
| 6  | Matrix Inversion Methods, Factorization(Doolittle, and Crout) Method and its various Forms + <b>Assignment#2</b>   |
| 7  | Factorization(Cholesky) Method and its various Forms,  |
| 8  | Iterative Methods: Jacobi method, Gauss-Seidel method, SOR Methods.  |
| 9  | <b>MID TERM</b> , Ill-Condition system and condition number  |
| 10 | Eigen values and Eigen Vectors, Power and Rayleigh Quotient method, <b>Assignment#3</b>  |
| 11 | Interpolating and Polynomial Approximation: Difference Operators, Interpolation with unequal intervals: Lagrange's Interpolation Formula. <b>QUIZ 3.</b> |
| 12 | Newton's Divided Difference Formula, Error in Polynomial Interpolation.  |
| 13 | Interpolation with equal intervals: Gregory Newton Forward/Backward Interpolation Formula, <b>QUIZ 4</b>   |
| 14 | Gauss's Forward/Backward Interpolation Formula, Stirling's Formula, Laplace Everett's Formula, <b>Assignment#4</b>                                       |
| 15 | Bessel's Formula, Aitken's Interpolation, Hermite Interpolation.   |
| 16 | Presentations  |



# NUMERICAL ANALYSIS

## \* Introduction:-

Most of mathematical problems can be solved by analytical methods, they give exact or true solution of problem but sometimes it is rather difficult to solve the problem by analytic method.

For example:-  $\int_0^1 \frac{e^{-x^2} x^2 \sin^2 x}{\sqrt{1+x^2}} dx$

Many such examples can be cited for which solution through analytical method is not possible or very complex that is not suitable according to our requirement. In this situation where analytical method does not work we use Numerical Methods.

Numerical methods give approximate solution of problem in steps or in series.

In numerical analysis we study to solve problems by various methods which can calculate the approximate solution of problem. As approximation of solution is near to exact solution then there is error in numerical problem.



## SECTION (I)

### • Error:-

An error in numerical problem, is difference between the exact value of expression and its computed value.

Let  $x$  be actual value and  $\bar{x}$  be computed value then error is  

$$\text{Error} = E = x - \bar{x}$$

### • SOURCE OF ERROR:-

The error that affect the mathematical computations may be divided into 5 types

#### (i) Input Error:-

The input data can be the result of measurements with limited accuracy or real number which must be represented with fixed numbers of digit.

Example: 2.56782

We write it as 2.567

#### (ii) Modelling Error:-

The discrepancy between the model and physical system is source of error which can be termed as error of problem.

Like friction, gravity etc.

#### (iii) Rounding Error:-

The error which arises when we use fixed number of digits in calculation.

If the calculating device cannot be handled off to  $t$ -digits, number can not be used in subsequent calculation. Then the <sup>resulting</sup> value must be rounding off.

#### (iv) Truncation Error:-

The error arises when an infinite process is replaced by a finite one.

For example:  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

We use only  $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$  to solve problem.

#### (v) Initial Error:-

It is also called human error. These errors can be expected in program itself, by operator.

Most of these errors can be minimized by proper planning.

## \* Types of Errors:-

### (i) Absolute Error:-

If  $x$  is the actual value and  $\bar{x}$  is approximate value then absolute error is defined by

$$E_a = A.E = |x - \bar{x}|$$

$$= |\Delta x|$$

Where  $\Delta x = x - \bar{x}$

Example:  $x = 4.83$ ,  $\bar{x} = 4.832$

$$E_a = |x - \bar{x}|$$

$$= 0.002$$

Note:-

Generally if a number is correct upto  $n$ -decimal places it has a rounding error

$$A.E \leq 1 \times 10^{-n}$$

(iii)

### (ii) Relative Error:-

Let  $x$  be true value of a quantity and  $\bar{x}$  be computed value then relative error is denoted and defined as

$$E_r = \frac{|x - \bar{x}|}{x} \quad \text{sit } x \neq 0$$

$$= \frac{|\Delta x|}{x}$$

$$\text{As } x \approx \bar{x}$$

$$\text{Then } E_r = \frac{|\Delta x|}{\bar{x}}$$

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As the measure of accuracy, relative error is more precise than absolute error. The size of absolute error depends on the unit, where as the relative is dimensionless quantity.

Note:-

A decimal number correct to  $n$ -significant digit has

$$R.E = E_r \leq 5 \times 10^{-n}$$

### (iii) Percentage Error:-

Relative error is expressed in percentage is called percentage error. It is denoted and defined as

$$P.E = E_p = \frac{R.E \times 100}{x}$$

$$= \frac{|x - \bar{x}| \times 100}{x}$$

Example:  $x = 4.83$ ,  $\bar{x} = 4.832$

$$E_a = 0.002$$

$$R.E = \frac{|x - \bar{x}|}{x} = \frac{0.002}{4.83} = 0.0041$$

$$P.E = R.E \times 100$$

$$= 0.0041 \times 100$$

$$= 0.41\%$$

It is also called probable error.

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## • Effect of Error in Arithmetic Operations:-

### (i) Error in Addition:-

Let  $\bar{x}_1$  and  $\bar{x}_2$  be approximate value of  $x_1$  and  $x_2$ . Also  $E_1$  be error in  $x_1$  and  $E_2$  be error in  $x_2$ . Then

$$E_1 = x_1 - \bar{x}_1, \quad E_2 = x_2 - \bar{x}_2$$

Let  $E$  be error of ' $x_1 + x_2$ '

$$\begin{aligned} E &= (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2) \\ &= x_1 + x_2 - \bar{x}_1 - \bar{x}_2 \\ &= E_1 + \bar{x}_1 + E_2 + \bar{x}_2 - \bar{x}_1 - \bar{x}_2 \\ &\because x_1 = E_1 + \bar{x}_1 \end{aligned}$$

$$E = E_1 + E_2$$

Taking absolute on both sides

$$|E| = |E_1 + E_2|$$

$$\Rightarrow |E| \leq |E_1| + |E_2|$$

which shows that absolute error of a sum of numbers is not greater than the sum of absolute errors of numbers. In general, For  $n$ -numbers

$$|E| \leq |E_1| + |E_2| + \dots + |E_n|$$

Hence the absolute error of the sum of  $n$ -numbers doesn't exceed the sum of absolute error of numbers.

### (ii) Error in Subtraction:-

Let  $\bar{x}_1$  and  $\bar{x}_2$  be approximate value of  $x_1$  and  $x_2$ . Also  $E_1$  and  $E_2$  be error in  $x_1$  and  $x_2$  resp. Let  $E$  be error in ' $x_1 - x_2$ '

$$\begin{aligned} E &= (x_1 - x_2) - (\bar{x}_1 - \bar{x}_2) \\ &= x_1 - x_2 - \bar{x}_1 + \bar{x}_2 \\ &= E_1 + \bar{x}_1 - E_2 - \bar{x}_2 - \bar{x}_1 + \bar{x}_2 \\ &= E_1 - E_2 \quad \because E_1 = x_1 - \bar{x}_1 \\ &= E_1 + (-E_2) \quad \because x_2 = E_2 + \bar{x}_2 \end{aligned}$$

$$|E| = |E_1 + (-E_2)|$$

$$\leq |E_1| + |-E_2|$$

$$|E| \leq |E_1| + |E_2|$$

Thus the absolute error of numbers is not greater than the sum of absolute error of the numbers.

### (iii) Error in Product:-

Let  $\bar{x}_1$  and  $\bar{x}_2$  be approximate value of  $x_1$  and  $x_2$ . Let  $E_1$  and  $E_2$  be errors in  $x_1$  and  $x_2$  respectively

$$E_1 = x_1 - \bar{x}_1$$

$$E_2 = x_2 - \bar{x}_2$$

Let  $E$  be error in ' $x_1 \cdot x_2$ '

$$\begin{aligned} E &= x_1 x_2 - \bar{x}_1 \bar{x}_2 \\ &= (E_1 + \bar{x}_1)(E_2 + \bar{x}_2) - \bar{x}_1 \bar{x}_2 \\ &= E_1 E_2 + E_1 \bar{x}_2 + E_2 \bar{x}_1 + \bar{x}_1 \bar{x}_2 - \bar{x}_1 \bar{x}_2 \end{aligned}$$



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$$= E_1(x_2 - E_2) + E_2(x_1 - E_1) + E_1 E_2$$

$$= E_1 x_2 - E_1 E_2 + E_2 x_1 - E_1 E_2 + E_1 E_2$$

$$E = E_1 x_1 + E_2 x_1 - E_1 E_2$$

Dividing both sides by  $x_1 x_2$

$$\frac{E}{x_1 x_2} = \frac{E_1 x_2}{x_1 x_2} + \frac{E_2 x_1}{x_1 x_2} - \frac{E_1 E_2}{x_1 x_2}$$

$$= \frac{E_1}{x_1} + \frac{E_2}{x_2} - \frac{E_1 E_2}{x_1 x_2}$$

As  $\frac{E_1 E_2}{x_1 x_2}$  is very small quantity so neglecting

$$\Rightarrow \frac{E}{x_1 x_2} \approx \frac{E_1}{x_1} + \frac{E_2}{x_2}$$

Taking Absolute both sides

$$\left| \frac{E}{x_1 x_2} \right| \approx \left| \frac{E_1}{x_1} + \frac{E_2}{x_2} \right|$$

$$\leq \left| \frac{E_1}{x_1} \right| + \left| \frac{E_2}{x_2} \right|$$

Thus the relative error of product is approximately the sum of relative error of factors.

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(iv) Error in Division:-

Let  $\bar{x}_1$  and  $\bar{x}_2$  be approximate values of  $x_1$  and  $x_2$ . Let  $E_1$  and  $E_2$  be errors of  $x_1$  and  $x_2$  resp. Let  $E$  be error in  $x_1/x_2$ .

$$E = \frac{x_1}{x_2} - \frac{\bar{x}_1}{\bar{x}_2}$$

$$= \frac{x_1}{x_2} - \frac{(x_1 - E_1)}{(x_2 - E_2)} \quad \because \bar{x}_1 = x_1 - E_1$$

$$= \frac{x_1}{x_2} - \frac{(x_1 - E_1)(x_2 - E_2)^{-1}}{x_2}$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} \left( \frac{1 - E_1}{x_1} \right) \left( \frac{1 - E_2}{x_2} \right)^{-1} x_2^{-1}$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} \left( \frac{1 - E_1}{x_1} \right) \left( \frac{1 - E_2}{x_2} \right)^{-1}$$

$\therefore$  By binomial expansion

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} \left( \frac{1 - E_1}{x_1} \right) \left( 1 + \frac{E_2}{x_2} + \dots \right)$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} \left( \frac{1 - E_1 + E_2 + \dots}{x_1} \right)$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} + \frac{E_1}{x_2} - \frac{E_2 x_1}{x_2^2} + \dots$$

Neglecting higher powers of  $E_1$  &  $E_2$

$$\frac{E}{x_1/x_2} = \frac{E_1}{x_2} - \frac{E_2 x_1}{x_2^2}$$

$$\left| \frac{x_2 E}{x_1} \right| = \left| \frac{E_1}{x_1} + \left( \frac{-E_2}{x_2} \right) \right|$$

$$\leq \left| \frac{E_2}{x_2} \right| + \left| \frac{E_1}{x_1} \right|$$

(proved).

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## (v) Error in Function of n-Variables:-

Let

$y = f(x_1, x_2, \dots, x_n)$  be a function of  $n$ -variable and let  $\delta x_1, \delta x_2, \dots, \delta x_n$  be error in  $x_1, x_2, \dots, x_n$  and  $\delta y$  be error in  $y$ .

Taking increment on both sides

$$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2, \dots, x_n + \delta x_n)$$

Now,

Using Taylor's theorem

$$y + \delta y = f(x_1, x_2, \dots, x_n) + \left( \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n} \right) + \dots$$

Neglecting square terms of  $\delta x_1, \delta x_2, \dots, \delta x_n$  we have

$$y + \delta y = y + \left( \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n} \right)$$

$$\delta y = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n}$$

we get change in  $y$  w.r.t  $n$ -Variables.  
For absolute error we taking absolute on both sides we get

$$|\delta y| = \left| \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n} \right|$$

$$|\delta y| \leq \left| \frac{\partial f}{\partial x_1} \right| |\delta x_1| + \left| \frac{\partial f}{\partial x_2} \right| |\delta x_2| + \dots + \left| \frac{\partial f}{\partial x_n} \right| |\delta x_n|$$

## \* Question:-

Find the error in product of  $n$ -numbers?

## Solution:-

Let  $y = x_1 x_2 x_3 \dots x_n \rightarrow (i)$   
and  $\delta y$  be error in  $y$  then

$$\delta y = \frac{\partial y}{\partial x_1} \delta x_1 + \frac{\partial y}{\partial x_2} \delta x_2 + \dots + \frac{\partial y}{\partial x_n} \delta x_n \rightarrow (ii)$$

Taking Log on both sides of (i) we get  
 $\ln y = \ln x_1 + \ln x_2 + \dots + \ln x_n$

Differentiate w.r.t  $x_1$ 

$$\frac{1}{y} \frac{\partial y}{\partial x_1} = \frac{1}{x_1}$$

$$\Rightarrow \frac{\partial y}{\partial x_1} = \frac{y}{x_1}$$

$$\text{Similarly; } \frac{\partial y}{\partial x_2} = \frac{y}{x_2}$$

$$\dots \dots \dots \frac{\partial y}{\partial x_n} = \frac{y}{x_n}$$

Put in eq. (ii)

$$\delta y = \frac{y}{x_1} \delta x_1 + \frac{y}{x_2} \delta x_2 + \dots + \frac{y}{x_n} \delta x_n$$

For relative error we divide by 'y'

$$\frac{\delta y}{y} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \dots + \frac{\delta x_n}{x_n}$$

For absolute Error



$$\left| \frac{\delta y}{y} \right| \leq \left| \frac{\delta x_1}{x_1} \right| + \left| \frac{\delta x_2}{x_2} \right| + \dots + \left| \frac{\delta x_n}{x_n} \right|$$

$$E_y \leq E_{1x} + E_{2x} + \dots + E_{nx}$$

Thus relative error of product of  $n$ -number cannot exceed the algebraic sum of there relative error.

\* Question:-

Find relative error of  $y$  where  
 $y = a x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$

Solution:-

Let  $\delta y$  be error in  $y$

$$\delta y = \frac{\partial f}{\partial x_1} \delta x_1 + \frac{\partial f}{\partial x_2} \delta x_2 + \dots + \frac{\partial f}{\partial x_n} \delta x_n \quad (*)$$

As  $y = a x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$

taking log on both sides

$$\ln y = \ln a + m_1 \ln x_1 + m_2 \ln x_2 + \dots + m_n \ln x_n$$

Taking partial derivative w.r.t  $x_1$

$$\frac{\partial y}{\partial x_1} = \frac{m_1 y}{x_1}$$

$$\frac{\partial y}{\partial x_1} = y \frac{m_1}{x_1}$$

Similarly,  $\frac{\partial y}{\partial x_2} = y \frac{m_2}{x_2}$

$$\vdots$$

$$\frac{\partial y}{\partial x_n} = y \frac{m_n}{x_n}$$

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(\*)  $\Rightarrow$

$$\delta y = y \frac{m_1}{x_1} \delta x_1 + y \frac{m_2}{x_2} \delta x_2 + \dots + y \frac{m_n}{x_n} \delta x_n$$

$$\left| \frac{\delta y}{y} \right| = \left| \frac{m_1}{x_1} \delta x_1 + \frac{m_2}{x_2} \delta x_2 + \dots + \frac{m_n}{x_n} \delta x_n \right|$$

$$\leq \left| \frac{m_1}{x_1} \right| |\delta x_1| + \left| \frac{m_2}{x_2} \right| |\delta x_2| + \dots + \left| \frac{m_n}{x_n} \right| |\delta x_n|$$

$$E_y \leq m_1 E_{1x} + m_2 E_{2x} + \dots + m_n E_{nx}$$

Thus the relative error of  $y$  cannot be exceed algebraic sum of there relative error.

\* Question:- Find relative error of  $y = x^m$

Solution:- Let  $\delta y$  be error in  $y$

$$\Rightarrow \delta y = \frac{\partial y}{\partial x} \delta x \quad (i)$$

As  $y = x^m$

$$\ln y = m \ln x$$

$$\frac{\partial y}{\partial x} = m \frac{y}{x}$$

$$\frac{\partial y}{\partial x} = \frac{m y}{x}$$

Put in (i),  $\delta y = \frac{m y}{x} \delta x$

$$\frac{\delta y}{y} = \frac{m y}{y} \frac{\delta x}{x}$$

$$\Rightarrow \left| \frac{\delta y}{y} \right| = m \left| \frac{\delta x}{x} \right|$$

$$\Rightarrow E = m E_x \quad (\text{Ans})$$

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## DEFINITION:- (Significant Figures)

Significant figures (or digits) of an approximate number is any of the digit 1, 2, 3, ..., 9 in its decimal representation, or any zero except when it is used to fix the decimal point or to fill the place of unknown or discarded digit.

Example:

(i) Number 0.00475 the significant figures are 4, 7, 5. First three zeros are used merely to fix the position of decimal point and indicate the place values of other digits and are therefore not significant.

(ii) Number 8309, all digits, including zero, are significant.

(iii) In number 0.0006070, the first four zeros are not significant. All remaining digits, including the other two zeros, are significant.

## ROUNDING OFF RULE:-

To round off or simply round a number to  $n$  significant figures or digits, drop all digits to right of  $n^{\text{th}}$  place, or replace them by zeros (0) if the zeros are needed as place holder. In this rounding-off rule, note following:

- If the first of discarded digits is less than 5, leave remaining digits unchanged.
- If first discard digit is exactly 5 and there are non-zero digits among those discarded, add 1 to last retained digit.
- If first discarded digit exceeds 5, add 1 to the last retained digit.
- If first discarded digit, however is exactly 5 and all other discarded digits are zeros, the last retained digit is left unchanged or is increased by 1 according as it is even or odd (the even-digit rule).

# The Solution Of Equations In One Variable

(Linear, Non Linear, Algebraic, Transcendental)

The expression of the form  
 $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$   
 where  $a_0 \neq 0$ ,  $a_i$  is constant and  $n$  is positive integer, is called a polynomial of degree  $n$  in variable  $x$ .

$f(x) = 0$  is called an algebraic equation

Example:  $2x^3 - x + 10 = 0$  is algebraic equation.

An equation which contains some other function such as trigonometric, logarithmic, exponential are called transcendental equation.

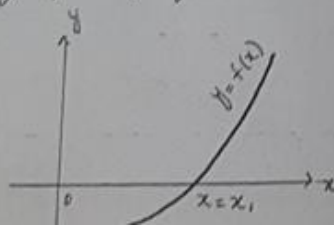
Example:  $xe^x - \cos 2x = 0$   
 $x + \ln x = 0$ .

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The value of  $x$  which satisfies the equation  $f(x) = 0$  is called root or solution of equation.

Geometrically, a root of  $f(x) = 0$  is that value of  $x$  where the graph of  $y = f(x)$  crosses  $x$ -axis

From fig  $x = x_1$  is a root of equation  $f(x) = 0$ .



\* Intermediate Value Property :- If  $f(x)$  is continuous on  $[a, b]$  and  $f(a)$  and  $f(b)$  have opposite sign then  $f(x) = 0$  has at least one real root between 'a' and 'b'.

## \* METHODS FOR SOLVING EQUATIONS IN ONE VARIABLE

We shall discuss the following methods for solving equations in one variable.

1. Bisection Method (OR)  
Interval halving Method.
2. Regula Falsi Method (OR)  
Method of False Position
3. Newton-Raphson Method.
4. Iteration Method (OR)  
Fixed Point Iteration.
5. Secant Method.

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(1) Bisection Method (or)  
Interval Halving Method.

In order to solve  $f(x)=0$  Consider  $f(x)$  is continuous on  $[a, b]$  and having opposite sign at  $f(a)$  and  $f(b)$ .

i.e.  $f(a) \cdot f(b) < 0$

Then by intermediate value property there is a real root  $\alpha$  of  $f(x)=0$  in  $[a, b]$ .

Let  $c_1 = \frac{a+b}{2}$  be 1<sup>st</sup> approximation to the root.

If  $f(c_1)=0$ , then  $c_1$  is required root.

If  $f(c_1) \neq 0$  and

if  $f(a)f(c_1) < 0$ , then root lies in  $]a, c_1[$

if  $f(c_1)f(b) < 0$ , then root lies in  $]c_1, b[$ .

Suppose root lies in  $]c_1, b[$  then we find the mid point  $c_2$  of  $]c_1, b[$

i.e.  $c_2 = \frac{c_1+b}{2}$  and find  $f(c_2)$ .

If  $f(c_2)=0$ , then  $c_2$  is required root.

If  $f(c_2) \neq 0$  and

if  $f(c_1)f(c_2) < 0$ , then root lies in  $]c_1, c_2[$

if  $f(c_2)f(b) < 0$ , then root lies in  $]c_2, b[$

By having the intervals repeatedly we can find an approximation to the exact root of  $f(x)=0$ .



Example:-Solve equation  $x^3 + x - 1 = 0$  using Bisection method.Solution:-

Let  $f(x) = x^3 + x - 1$

|        |    |        |        |        |       |
|--------|----|--------|--------|--------|-------|
| $x$    | 0  | 0.2    | 0.4    | 0.6    | 0.8   |
| $f(x)$ | -1 | -0.792 | -0.536 | -0.184 | 0.312 |

As sign change between 0.6 and 0.8

i.e.  $f(0.6) = -0.184 < 0$

$f(0.8) = 0.312 > 0$

As  $f(0.6) \cdot f(0.8) < 0$ , so root lies between  $]0.6, 0.8[$ .1<sup>st</sup> Iteration:

Let  $a_1 = 0.6$ ,  $b_1 = 0.8$ ,  $c_1 = \frac{a+b}{2}$

$\Rightarrow c_1 = \frac{0.6 + 0.8}{2} = 0.700$

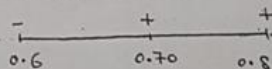
$f(c_1) = f(0.700)$

$= (0.700)^3 + (0.700) - 1$

$= 0.0430$

$f(0.6) \cdot f(0.700) < 0$

So root lies

between  $]0.6, 0.70[$ .

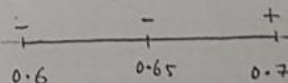
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2<sup>nd</sup> Iteration:

$a_1 = 0.6$ ,  $b_1 = 0.700$

$c_2 = \frac{0.6 + 0.700}{2} = 0.6500$

$f(c_2) = -0.0754$



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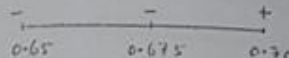
$$f(0.6500)f(0.700) < 0$$

So root lies between  $]0.65, 0.70[$ .

3<sup>rd</sup> Iteration:

$$\text{Let } a_3 = 0.6500, b_3 = 0.700$$

$$c_3 = \frac{0.6500 + 0.700}{2} = 0.6750$$

$$f(c_3) = (0.6750)^3 + (0.6750) - 1$$


$$= -0.0175$$

$$\text{As } f(0.6750)f(0.70) < 0$$

So root lies between  $]0.6750, 0.70[$ .

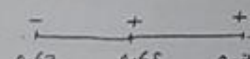
4<sup>th</sup> Iteration:

$$\text{Let } a_4 = 0.6750, b_4 = 0.700$$

$$c_4 = \frac{a_4 + b_4}{2} = \frac{0.6750 + 0.700}{2}$$

$$\Rightarrow c_4 = 0.6875$$

$$f(c_4) = (0.6875)^3 + (0.6875) - 1$$

$$= 0.0125$$


$$\text{As } f(0.6750)f(0.6875) < 0$$

So root lies between  $]0.6750, 0.6875[$ .

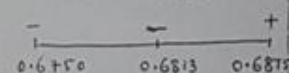
5<sup>th</sup> Iteration:

$$a_5 = 0.6750, b_5 = 0.6875$$

$$c_5 = \frac{0.6750 + 0.6875}{2}$$

$$c_5 = 0.6813$$

$$f(c_5) = (0.6813)^3 + (0.6813) - 1$$

$$= -0.0025$$


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$$\text{As } f(0.6813)f(0.6875) < 0$$

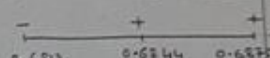
So root lies between  $]0.6813, 0.6875[$ .

6<sup>th</sup> Iteration:

$$a_6 = 0.6813, b_6 = 0.6875$$

$$c_6 = \frac{0.6813 + 0.6875}{2} = 0.6844$$

$$f(c_6) = (0.6844)^3 + (0.6844) - 1$$

$$= 0.0050$$


$$\text{As } f(0.6813)f(0.6844) < 0$$

So root lies between  $]0.6813, 0.6844[$ .

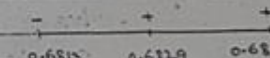
7<sup>th</sup> Iteration:

$$a_7 = 0.6813, b_7 = 0.6844$$

$$c_7 = \frac{0.6813 + 0.6844}{2}$$

$$c_7 = 0.6829$$

$$f(c_7) = (0.6829)^3 + (0.6829) - 1$$

$$= 0.0014$$


$$\text{As } f(0.6813)f(0.6829) < 0$$

So root lies between  $]0.6813, 0.6829[$ .

8<sup>th</sup> Iteration:

$$a_8 = 0.6813, b_8 = 0.6829$$

$$c_8 = \frac{0.6813 + 0.6829}{2}$$

$$c_8 = 0.6821$$

$$f(c_8) = (0.6821)^3 + (0.6821) - 2$$

$$= -0.0005$$

$$\text{As } f(0.6821)f(0.6829) < 0$$

So root lies between  $]0.6821, 0.6829[$

9<sup>th</sup> Iteration:

$$a_9 = 0.6821, b_9 = 0.6829$$

$$c_9 = \frac{0.6821 + 0.6829}{2}$$

$$c_9 = 0.6825$$

$$f(c_9) = (0.6825)^3 + (0.6825) - 2$$

$$= 0.0004$$

So approximate root is 0.6825

Example:-

Solve  $f(x) = \sin x - 5x + 2$  by  
Bisection method.

Solution: As  $f(x) = \sin x - 5x + 2$

| x    | 0 | 0.1    | 0.2    | 0.3    | 0.4    | 0.5   | 0.6    |
|------|---|--------|--------|--------|--------|-------|--------|
| f(x) | 2 | 1.5998 | 1.1986 | 0.7955 | 0.3894 | -0.02 | -0.435 |

As  $f(x)$  changes sign between (0.4, 0.5)

$$\text{i.e. } f(0.4) = 0.3894 > 0$$

$$f(0.5) = -0.02 < 0$$

$$f(0.4)f(0.5) < 0$$

So root lies between  $]0.4, 0.5[$

1<sup>st</sup> Iteration:  $a_1 = 0.4, b_1 = 0.5$

$$c_1 = \frac{0.4 + 0.5}{2} = 0.4500$$

$$f(c_1) = \sin(0.4500) - 5(0.4500) + 2$$

$$= -0.1850$$

$$f(0.45)f(0.5) < 0$$

So root lies between  $]0.4500, 0.5[$

2<sup>nd</sup> Iteration:

$$a_2 = 0.4500, b_2 = 0.5$$

$$c_2 = \frac{0.4500 + 0.5}{2} = 0.4750$$

$$f(c_2) = \sin(0.4750) - 5(0.4750) + 2$$

$$= 0.0823$$

$$\text{As } f(0.4750)f(0.5) < 0$$

So root lies between  $]0.4750, 0.5[$

3<sup>rd</sup> Iteration:  $a_3 = 0.4750, b_3 = 0.5$

$$c_3 = \frac{0.4750 + 0.5}{2} = 0.4875$$

$$f(c_3) = f(0.4875)$$

$$= \sin(0.4875) - 5(0.4875) + 2$$

$$= 0.0309$$

$$f(0.4875)f(0.5) < 0$$

So root lies between  $]0.4875, 0.5[$

4<sup>th</sup> Iteration:

$$a_4 = 0.4875, b_4 = 0.5$$

$$c_4 = \frac{0.4875 + 0.5}{2} = 0.4938$$



$$f(c_4) = f(0.4938) \\ = \sin(0.4938) - 5(0.4938) + 5 \\ = 0.0050$$

As  $f(0.4938) \cdot f(0.5) < 0$   
So root lies between  $]0.4938, 0.5[$ .

5<sup>th</sup> Iteration:

$$a_5 = 0.4938, b_5 = 0.5 \\ c_5 = \frac{0.4938 + 0.5}{2} = 0.4969$$

$$f(c_5) = f(0.4969) \\ = \sin(0.4969) - 5(0.4969) + 5 \\ = -0.0078$$

As  $f(0.4969) \cdot f(0.4938) < 0$   
So root lies between  $]0.4938, 0.4969[$ .

6<sup>th</sup> Iteration:

$$a_6 = 0.4938, b_6 = 0.4969 \\ c_6 = \frac{0.4938 + 0.4969}{2} = 0.4954$$

$$f(c_6) = f(0.4954) \\ = \sin(0.4954) - 5(0.4954) + 5 \\ = -0.0016$$

As  $f(0.4938) \cdot f(0.4954) < 0$   
So root lies between  $]0.4938, 0.4954[$ .

7<sup>th</sup> Iteration:

$$a_7 = 0.4938 \\ b_7 = 0.4954$$

$$c_7 = \frac{0.4938 + 0.4954}{2} = 0.4946$$

$$f(c_7) = f(0.4946) \\ = \sin(0.4946) - 5(0.4946) + 5 \\ = 0.4946$$

As  $f(0.4946) \cdot f(0.4954) < 0$   
So root lies between  $]0.4946, 0.4954[$ .

8<sup>th</sup> Iteration:

$$a_8 = 0.4946 \\ b_8 = 0.4954 \\ c_8 = \frac{0.4946 + 0.4954}{2} = 0.4950$$

$$f(c_8) = f(0.4950) \\ = \sin(0.4950) - 5(0.4950) + 5 \\ = 0.0000$$

So approximate root is 0.4950.

\* Example:

Use Bisection method to find the root of  $2^x - 5x + 2 = 0$ , correct upto two decimal places.

Solution:

$$\text{Let } f(x) = 2^x - 5x + 2$$

|        |   |    |
|--------|---|----|
| $x$    | 0 | 1  |
| $f(x)$ | 3 | -1 |

$$\text{As } f(0) \cdot f(1) < 0$$

So root lies between  $]0, 1[$ .

1<sup>st</sup> Iteration:

$$a_1 = 0, b_1 = 1, c_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5$$

Now

$$\begin{aligned} f(c_1) &= f(0.5) \\ &= (2)^{(0.5)} - 5(0.5) + 2 \\ &= 0.91 \end{aligned}$$

As  $f(0.5)f(1) < 0$  so root lies between  $]0.5, 1[$ .

2<sup>nd</sup> Iteration:

$$a_2 = 0.5, b_2 = 1$$

$$c_2 = \frac{0.5 + 1}{2} = 0.75$$

$$\begin{aligned} f(c_2) &= f(0.75) \\ &= (2)^{(0.75)} - 5(0.75) + 2 \\ &= -0.07 \end{aligned}$$

As  $f(0.5)f(0.75) < 0$   
so root lies between  $]0.5, 0.75[$ .

3<sup>rd</sup> Iteration:

$$a_3 = 0.5, b_3 = 0.75$$

$$c_3 = \frac{0.5 + 0.75}{2} = 0.63$$

$$\begin{aligned} f(c_3) &= f(0.63) \\ &= (2)^{(0.63)} - 5(0.63) + 2 \\ &= 0.39 \end{aligned}$$

As  $f(0.63)f(0.75) < 0$ , so root lies between  $]0.63, 0.75[$ .

4<sup>th</sup> Iteration:

$$a_4 = 0.63, b_4 = 0.75$$

$$c_4 = \frac{0.63 + 0.75}{2} = 0.70$$

$$\begin{aligned} f(c_4) &= f(0.70) \\ &= (2)^{(0.70)} - 5(0.70) + 2 \\ &= 0.12 \end{aligned}$$

As  $f(0.70)f(0.75) < 0$ , so root lies between  $]0.70, 0.75[$ .

5<sup>th</sup> Iteration:

$$a_5 = 0.70, b_5 = 0.75$$

$$c_5 = \frac{0.70 + 0.75}{2} = 0.73$$

$$\begin{aligned} f(c_5) &= f(0.73) \\ &= (2)^{(0.73)} - 5(0.73) + 2 \\ &= 0.01 \end{aligned}$$

As  $f(0.73)f(0.75) < 0$ , so root lies between  $]0.73, 0.75[$ .

6<sup>th</sup> Iteration:

$$a_6 = 0.73, b_6 = 0.75$$

$$c_6 = \frac{0.73 + 0.75}{2} = 0.74$$

$$\begin{aligned} f(c_6) &= f(0.74) = (2)^{(0.74)} - 5(0.74) + 2 \\ &= -0.03 \end{aligned}$$

$\Rightarrow$  Correct upto two decimal places  
required root is 0.73.

## Exercise

- 1- Use Bisection Method to Solve

(i)  $2e^{-x} - \sin x = 0$

Ans: 0.9210

(2)  $2x^3 + x - 2 = 0$

Ans: 0.4836

(3)  $x + x^x = 100$

Ans: 3.5812

- 2- Given that  $x = 1.45$  is exact solution of equation  $x^2 - 5.57x + 5.974 = 0$ . Apply 3 iterations of Bisection method for solving equation.

- 3- Find the root of equation  $2x - \cos x - 3 = 0$  upto 3 decimal places by bisection method.

Ans: 1.5235.

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## 2- Regula Falsi METHOD (OR) Method of False Position.

If  $f(a)f(b) < 0$  and  $f$  is continuous on  $[a, b]$ , then an approximation to the root of  $f(x) = 0$  on  $[a, b]$  is given by

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Proof:-

Let  $f(x)$  be a continuous function on  $[a_0, b_0]$ . Let  $y = f(x) = 0$  to solve equation by using regula falsi method we find approximate root by draw chord.

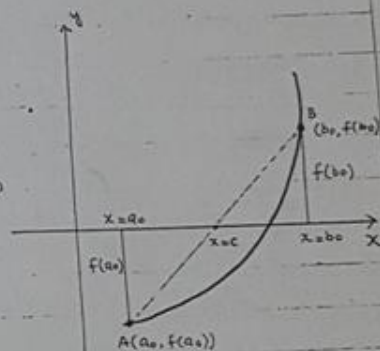
$$f(a_0) < 0, f(b_0) > 0$$

Let  $A(a_0, f(a_0))$  and  $B(b_0, f(b_0))$  be two points on a graph  $y = f(x)$ .

Draw a chord AB.

$$\text{Slope of line AB is } = \frac{f(b_0) - f(a_0)}{b_0 - a_0}$$

and equation of line through A and B is given by





$$(y - f(b_0)) = \frac{f(b_0) - f(a_0)}{b_0 - a_0} (x - b_0)$$

As  $x=c$  lies on  $x$ -axis at line AB  $(c, 0)$  so it satisfies that equation

$$0 - f(b_0) = \frac{f(b_0) - f(a_0)}{b_0 - a_0} (c - b_0)$$

$$-f(b_0)(b_0 - a_0) = (f(b_0) - f(a_0))(c - b_0)$$

$$-f(b_0)b_0 + f(b_0)a_0 = cf(b_0) - cf(a_0) - b_0f(b_0) + b_0f(a_0)$$

$$\Rightarrow a_0 f(b_0) - b_0 f(a_0) = c(f(b_0) - f(a_0))$$

$$\Rightarrow a_0 f(b_0) - b_0 f(a_0) = c(f(b_0) - f(a_0))$$

$$\Rightarrow \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)} = c$$

(i) If  $f(c) = 0$ , then we have exact root

(ii) If  $f(c) \neq 0$

Then we consider the intervals

$[a_0, c]$  s.t.  $f(a_0)f(c) < 0$

(OR)  $[c, b_0]$  s.t.  $f(c) \cdot f(b_0) < 0$

We continue this process and makes new intervals in general for  $[a_n, b_n]$  until we have exact root of equation.

$$\Rightarrow C = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

where  $n = 0, 1, 2, \dots$

1) Example:-Solve  $x \log_{10} x - 1.2 = 0$  correct upto 4 decimal places.Solution:

Let  $f(x) = x \log_{10} x - 1.2$

|        |      |         |        |
|--------|------|---------|--------|
| $x$    | 1    | 2       | 3      |
| $f(x)$ | -1.2 | -0.5979 | 0.2314 |

As  $f(2) \cdot f(3) < 0$  so root lies between 2 and 3.1<sup>st</sup> Iteration:

Let  $a_1 = 2$ ,  $b_1 = 3$

$f(a_1) = -0.5979$ ,  $f(b_1) = 0.2314$

$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 + 0.5979}$$

$$= 2.7210$$

$$f(c_1) = f(2.7210)$$

$$= (2.7210) \log_{10} (2.7210) - 1.2$$

$$= -0.0171$$

As  $f(2.7210) \cdot f(3) < 0$ 

So root lies between 2.7210 and 3.

2<sup>nd</sup> Iteration:

$$a_2 = 2.7210, f(a_2) = -0.0171$$

$$b_2 = 3, f(b_2) = 0.2314$$

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$$\Rightarrow C_2 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{2.7210 (0.2314) - 3(-0.0171)}{0.2314 + 0.0171}$$

$$= 2.7400$$

$$f(C_2) = f(2.7400)$$

$$= (2.7400) \log_{10}(2.7400) - 1.2$$

$$= -0.0006$$

$$\text{As } f(2.7400)f(3) < 0$$

So root lies between 2.7400 and 3.

3<sup>rd</sup> Iteration:

$$a_3 = 2.7400, \quad b_3 = 3$$

$$f(a_3) = -0.0006, \quad f(b_3) = 0.2314$$

$$C_3 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

$$= \frac{2.7400 (0.2314) - 3(-0.0006)}{0.2314 + 0.0006}$$

$$= 2.7405$$

$$f(C_3) = f(2.7405)$$

$$= (2.7405) \log_{10}(2.7405) - 1.2$$

$$= -0.0001$$

$$\text{As } f(2.7405)f(3) < 0$$

So root lies between 2.7405 and 3.

4<sup>th</sup> Iteration:

$$a_4 = 2.7405, \quad b_4 = 3$$

$$f(a_4) = -0.0001, \quad f(b_4) = 0.2314$$

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$$C_4 = \frac{a_4 f(b_4) - b_4 f(a_4)}{f(b_4) - f(a_4)}$$

$$= \frac{2.7405 (0.2314) - 3(-0.0001)}{0.2314 + 0.0001}$$

$$= 2.7408$$

$$f(C_4) = f(2.7408)$$

$$= (2.7408) \log_{10}(2.7408) - 1.2$$

$$= +0.0001$$

$$\text{As } f(2.7405)f(2.7408) < 0$$

So root lies between 2.7405 and 2.7408

5<sup>th</sup> Iteration:

$$a_5 = 2.7405, \quad b_5 = 2.7408$$

$$f(a_5) = -0.0001, \quad f(b_5) = 0.0001$$

$$\Rightarrow C_5 = \frac{a_5 f(b_5) - b_5 f(a_5)}{f(b_5) - f(a_5)}$$

$$= \frac{2.7405 (0.0001) - 2.7408 (-0.0001)}{0.0001 + 0.0001}$$

$$= 2.7406$$

$$f(C_5) = f(2.7406)$$

$$= (2.7406) \log_{10}(2.7406) - 1.2$$

$$= 0.0000$$

$\Rightarrow 2.7406$  is the required root

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 Ph: 4421848, 0333-6128231



\* Example: Solve equation  $x - e^{-x} = 0$  by regula falsi method? Given root lies between  $(0.5, 0.6)$ .

Solution:-

$$\text{Let } f(x) = x - e^{-x}$$

As given  $a_0 = 0.5$  ,  $b_0 = 0.6$   
 $f(a_0) = -0.1065$  ,  $f(b_0) = 0.0512$

1<sup>st</sup> Iteration:

$$c_0 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$

$$= \frac{0.5(0.0512) - 0.6(-0.1065)}{0.0512 + 0.1065}$$

$$c_0 = 0.5675$$

$$f(c_0) = (0.5675) - e^{-(0.5675)}$$

$$= 0.006$$

$$f(0.5)f(0.5675) < 0$$

So root lies between  $(0.5, 0.5675)$

2<sup>nd</sup> Iteration:

$$a = 0.5, \quad b = 0.5675$$

$$f(a) = -0.1065, \quad f(b) = 0.006$$

$$c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{0.5(0.006) - 0.5675(-0.1065)}{0.006 + 0.1065}$$

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$$c = 0.5671$$

$$f(c) = f(0.5671) - (0.5671)$$

$$= (0.5671) - e$$

$$= 0.0000$$

So required root is 0.5671.  
 (Answer)

Example:-

Find Cube root of 5 using method of false position upto 3-decimal places.

Solution :- As  $x = (5)^{1/3}$   
 $\Rightarrow x^3 = 5$   
 $\Rightarrow x^3 - 5 = 0$   
 $\Rightarrow f(x) = x^3 - 5$

|      |    |        |        |        |       |   |
|------|----|--------|--------|--------|-------|---|
| x    | 1  | 1.2    | 1.4    | 1.6    | 1.8   | 2 |
| f(x) | -4 | -3.272 | -2.256 | -0.904 | 0.832 | 3 |

$$\text{As } f(1.6)f(1.8) < 0$$

So root lies between  $(1.6, 1.8)$

1<sup>st</sup> Iteration:-

$$a_1 = 1.6, \quad b_1 = 1.8$$

$$f(a_1) = -0.904, \quad f(b_1) = 0.832$$

$$c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{1.6(0.832) - 1.8(-0.904)}{0.832 + 0.904}$$

$$= 1.741$$

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$$f(c_1) = f(1.7041) \\ = (1.7041)^3 - 5 \\ = -0.0514$$

$$As f(1.8) f(1.7041) < 0$$

So root lies between 1.8 and 1.7041.

2<sup>nd</sup> Iteration:

$$a_2 = 1.7041, \quad b_2 = 1.8$$

$$f(a_2) = -0.0514, \quad f(b_2) = 0.832$$

$$c_2 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{1.7041(0.832) - (1.8)(-0.0514)}{0.832 + 0.0514}$$

$$= 1.709$$

$$f(c_2) = f(1.709) \\ = (1.709)^3 - 5 \\ = -0.009$$

$$f(1.709) f(1.8) < 0$$

So root lies between 1.709 and 1.8.

3<sup>rd</sup> Iteration:

$$a_3 = 1.709$$

$$b_3 = 1.8$$

$$f(a_3) = -0.009$$

$$f(b_3) = 0.832$$

$$c_3 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

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$$c_3 = \frac{1.709(0.832) - (1.8)(-0.009)}{0.832 + 0.009} \\ = 1.710$$

$$f(c_3) = f(1.710) \\ = (1.710)^3 - 5 \\ = 0.000$$

required root is 1.710

(Answer)

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Ph: 4421649, 0333-612

# EXERCISE

1. Solve by Regula falsi Method

(i)  $3x - \cos x - 1 = 0$

Ans: 0.6071

(ii)  $e^{-x} - \sin\left(\frac{\pi x}{2}\right) = 0$  ( $\because x_2 = 90$ )

Ans: 0.4436.

2. Apply Regula falsi method to find approximate root of  $\cos x - x = 0$  correct upto 4 decimal places on interval  $(0.5, \pi/4)$ .

Ans: 0.7391 ( $\because x_1 = 0.7354$ )

3. Given that  $x = 1.45$  is exact root of equation  $x^2 - 5.57x + 5.774 = 0$ . Apply three iteration of Regula falsi method for solving equation.

4. Approximate the solution to within  $10^{-5}$  of following

a)  $3x^2 - e^x = 0$  Ans: 0.92001

b)  $x^2 + 10\cos x = 0$  Ans: 1.96887

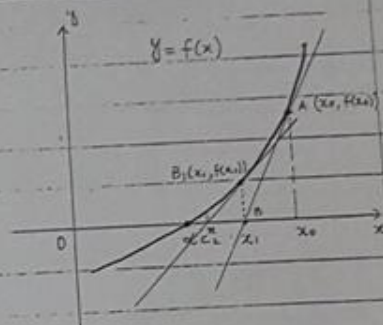
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# NEWTON-RAPHSON METHOD

Let  $\alpha$  be exact root of equation  $f(x) = 0$  then the approximate root is given

by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Proof:-

Let  $f(x) = 0$  be a function cuts the  $x$ -axis at  $\alpha$ , so  $\alpha$  be the exact root of equation. Let  $x_0$  be a initial guess. The point on curve  $A(x_0, f(x_0))$ .

Draw tangent at point A it cuts  $x$ -axis at point  $x_1$  say  $B(x_1, 0)$ .

Slope of tangent A is

$$\therefore \text{slope} = m = \frac{dy}{dx} \\ m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now point on curve is  $(x_1, f(x_1))$ .

Draw tangent at point  $B_1$  cut the



axis at point C so slope of B.C is

$$f'(x_1) = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Similarly;

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Continue this process until we have root of problem.

Therefore

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$n = 0, 1, 2, \dots$$

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Example:-

Use Newton raphson method to find root of equation  
 $\sin x = 1 - x$

Solution:-

$$\text{As } \sin x = 1 - x$$

$$\Rightarrow f(x) = \sin x + x - 1$$

|      |    |       |        |
|------|----|-------|--------|
| x    | 0  | 0.5   | 1      |
| f(x) | -1 | -0.02 | 0.8414 |

$$\text{As } f(x) = \sin x + x - 1$$

$$f(x_n) = \sin x_n + x_n - 1$$

$$f'(x_n) = \cos x_n + 1$$

Let initial guess root is 1 ( $x_0 = 1$ )

By Newton raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(\sin x_n + x_n - 1)}{\cos x_n + 1}$$

$$= \frac{x_n \cos x_n + x_n - \sin x_n - x_n + 1}{\cos x_n + 1}$$

$$x_{n+1} = \frac{x_n \cos x_n - \sin x_n + 1}{\cos x_n + 1} \rightarrow (*)$$

I<sup>st</sup> Iteration:

When  $n=0$  in (\*)

$$x_1 = \frac{x_0 \cos x_0 - \sin x_0 + 1}{\cos x_0 + 1}$$

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$$x_1 = \frac{1 \cos(1) - \sin(1) + 1}{\cos(1) + 1}$$

$$= 0.4537$$

2<sup>nd</sup> Iteration:When  $n=1$ 

$$x_2 = \frac{x_1 \cos x_1 - \sin x_1 + 1}{\cos x_1 + 1}$$

$$= \frac{(0.4537) \cos(0.4537) - \sin(0.4537) + 1}{\cos(0.4537) + 1}$$

$$= 0.5106$$

3<sup>rd</sup> Iteration:Put  $n=2$ 

$$x_3 = \frac{(0.5106) \cos(0.5106) - \sin(0.5106) + 1}{\cos(0.5106) + 1}$$

$$= 0.5110$$

4<sup>th</sup> Iteration:Put  $n=3$ 

$$x_4 = \frac{x_3 \cos x_3 - \sin x_3 + 1}{\cos x_3 + 1}$$

$$= \frac{(0.5110) \cos(0.5110) - \sin(0.5110) + 1}{\cos(0.5110) + 1}$$

$$x_4 = 0.5110$$

As  $x_3 = 0.5110$  and  $x_4 = 0.5110$ 

In Newton-Raphson method repeated root is called root or solution of equation.

$\Rightarrow$  Required root is 0.5110

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Example:-

Apply Newton Raphson Method to find solution of  $x^3 + x - 1 = 0$ .

Solution:-  $f(x) = x^3 + x - 1$ 

|        |    |    |   |
|--------|----|----|---|
| $x$    | -1 | 0  | 1 |
| $f(x)$ | -3 | -1 | 1 |

Root lies between (0, 1)

Let Guess root  $x_0 = 0$ 

$$As \quad f(x) = x^3 + x - 1$$

$$f'(x) = 3x^2 + 1$$

$$f'(x_n) = 3x_n^2 + 1$$

By Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^3 + x_n - 1)}{3x_n^2 + 1}$$

$$= \frac{3x_n^3 + x_n - x_n^3 - x_n + 1}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 1}{3x_n^2 + 1}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1} \rightarrow (i)$$

1<sup>st</sup> Iteration:Put  $n=0$  in (i)

$$x_1 = \frac{2x_0^3 + 1}{3x_0^2 + 1}$$

$$= \frac{1}{1} = 1$$

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2<sup>nd</sup> Iteration:Put  $n=1$  in (i)

$$x_2 = \frac{2x_1^3 + 1}{3x_1^2 + 1}$$

$$= \frac{2(1) + 1}{3(1) + 1} = \frac{3}{4} = 0.75$$

3<sup>rd</sup> Iteration:Put  $n=2$  in (i)

$$x_3 = \frac{2x_2^3 + 1}{3x_2^2 + 1}$$

$$= \frac{2(0.75)^3 + 1}{3(0.75)^2 + 1} = 0.6864$$

4<sup>th</sup> Iteration:Put  $n=3$  in (i)

$$x_4 = \frac{2x_3^3 + 1}{3x_3^2 + 1}$$

$$= \frac{2(0.6864)^3 + 1}{3(0.6864)^2 + 1} = 0.6823$$

5<sup>th</sup> Iteration:Put  $n=4$  in (i)

$$x_5 = \frac{2(0.6823)^3 + 1}{3(0.6823)^2 + 1}$$

$$x_5 = 0.6823$$

Hence the root is 0.6823.

(Answer).

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Example:-Using Newton's formula to find  $p^{\text{th}}$  root of  $N$ .Solution:-Let  $x$  be a  $p^{\text{th}}$  root of  $N$ 

$$\Rightarrow x = (N)^{1/p}$$

$$\Rightarrow x^p = N$$

$$\Rightarrow x^p - N = 0$$

$$\text{So } f(x) = x^p - N$$

$$f(x_n) = x_n^p - N$$

$$f'(x) = px^{p-1}$$

Now by Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^p - N)}{px_n^{p-1}}$$

$$= x_n - \frac{x_n^p - N}{px_n^{p-1}}$$

$$= \frac{px_n^p - x_n^p + N}{px_n^{p-1}}$$

$$= \frac{(p-1)x_n^p + N}{px_n^{p-1}}$$

$$\Rightarrow x_{n+1} = \frac{1}{p} \left[ (p-1)x_n + \frac{N}{x_n^{p-1}} \right]$$

(Answer)



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Example:

Show that Newton's method of Square root of  $N$  is given by  $x_{n+1} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)$  and Also find

Square root of 11 by above formula upto 4 - decimal places.

Solution:-

$$\text{Let } x = \sqrt{N}$$

$$x^2 = N$$

$$x^2 - N = 0$$

$$f(x) = x^2 - N$$

$$f(x_n) = x_n^2 - N$$

$$f'(x_n) = 2x_n$$

By Newton Raphson's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right) \rightarrow *$$

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Also when  $N = 11$

$$(*) \Rightarrow x_{n+1} = \frac{1}{2} \left( x_n + \frac{11}{x_n} \right) \rightarrow (*)$$

Initial guess is  $x_0 = 3$ .

Put  $n=0$  in  $*$

$$x_1 = \frac{1}{2} \left( 3 + \frac{11}{3} \right) = 3.3333$$

Put  $n=1$  in  $*$

$$x_2 = \frac{1}{2} \left( 3.3333 + \frac{11}{3.3333} \right) = 3.3167$$

Put  $n=2$  in  $*$

$$x_3 = \frac{1}{2} \left( 3.3167 + \frac{11}{3.3167} \right) = 3.3166$$

Put  $n=3$  in  $*$

$$x_4 = \frac{1}{2} \left( 3.3166 + \frac{11}{3.3166} \right) = 3.3166$$

So approximate root is 3.3166

(Answer)

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Exercise

- 1- By taking logarithm, find solution of  $x^x = 10$  correct to 4 decimal places by Newton-Raphson Method.

Ans: 2.5061

- 2- Using Newton-Raphson method, establish an iterative formula for finding a cube root of real number  $c$ .  
Also find For  $c = 30$  given that i.e.  $(30)^{1/3}$  initial guess is 3.

- 3- Find real root near to 1.5 of equation  $2x - 2\sin x - 1 = 0$ .

- 4- Find the negative root of equation  $e^x - \tan x = 0$  correct to 4 dp.

- 5- Using N-R method, evaluate a positive root of equation  $e^x - 3x = 0$  correct to 3 decimal places.

Ans: 1.5121

- 6- Use Newton Raphson Method to find the reciprocal of a number  $N$ .  
(Hint  $x = \frac{1}{N} \Rightarrow xN - 1 = 0$ )

## 4- Iteration Method (OR) Fixed Point Iteration

Definition:-

The process of finding successive approximation to a quantity is called iteration.

\* Working Rule:-

In order to solve  $f(x) = 0$  by fixed point iteration method, first guess the interval  $[a, b]$  s.t  $f(x) = 0$  has a root within interval.

Next write the equation as  $x = g(x)$  s.t on  $[a, b]$ .  
 $|g'(x)| < 1$  (convergence)

Then,

Setting the iteration as

$$x_{n+1} = g(x_n)$$

and by taking some initial guess  $x_0$  and also by applying successive iterations the root can be obtained.

\* Example:-

Find by fixed point iterative method to 5 dp root near 0.5 of equation  $\sin x = 5x - 2$ .

Solution:-

$$\text{Let } f(x) = \sin x - 5x + 2$$

|      |   |       |
|------|---|-------|
| x    | 0 | 0.5   |
| f(x) | 2 | -0.49 |

$\Rightarrow$  root lies between 0 and 0.5

Now,

$$\sin x = 5x - 2$$

$$\sin x + 2 = 5x$$

$$\Rightarrow x = \frac{1}{5} (\sin x + 2)$$

$$\Rightarrow g(x) = \frac{1}{5} (\sin x + 2)$$

$$g'(x) = \frac{1}{5} \cos x$$

$$|g'(x)| < 1 \quad \because |\cos x| < 1$$

so it is converges so setting iteration is

$$x_{n+1} = g(x_n)$$

$$\Rightarrow x_{n+1} = \frac{1}{5} (\sin x_n + 2) \rightarrow (i)$$

Let guess root is 0.4 be initial guess.

1<sup>st</sup> Iteration:Put  $n=0$  in (i)

$$\begin{aligned} x_1 &= \frac{1}{5} (2 + \sin x_0) \\ &= \frac{1}{5} (2 + \sin(0.4)) \\ &= 0.4779 \end{aligned}$$

2<sup>nd</sup> Iteration:

$$\begin{aligned} \text{Put } n=1 \text{ in (i)} \\ x_2 &= \frac{1}{5} (2 + \sin x_1) \\ &= \frac{1}{5} (2 + \sin(0.4779)) \\ &= 0.4919 \end{aligned}$$

3<sup>rd</sup> Iteration:

$$\begin{aligned} \text{Put } n=2 \text{ in (i)} \\ x_3 &= \frac{1}{5} (2 + \sin x_2) \\ &= \frac{1}{5} (2 + \sin(0.4919)) \\ &= 0.4945 \end{aligned}$$

4<sup>th</sup> Iteration:

$$\begin{aligned} \text{Put } n=3 \text{ in (i)} \\ x_4 &= \frac{1}{5} (2 + \sin x_3) \\ &= \frac{1}{5} (2 + \sin(0.4945)) \\ &= 0.4949 \end{aligned}$$

5<sup>th</sup> Iteration:

$$\begin{aligned} \text{Put } n=4 \text{ in (i)} \\ x_5 &= \frac{1}{5} (2 + \sin x_4) \\ &= \frac{1}{5} (2 + \sin(0.4949)) \\ &= 0.4949 \end{aligned}$$

So  $x = 0.4949$  is required root.

(Answer).



Example:-

Find by fixed point iterative method to find root near 0.5 of equation  $\sin x = 5x - 2$

Solution:-

Let  $f(x) = \sin x - 5x + 2$

|        |   |       |
|--------|---|-------|
| $x$    | 0 | 0.5   |
| $f(x)$ | 2 | -0.49 |

∴ root lies between 0 and 0.5

Now,

$$\sin x = 5x - 2$$

$$\sin x + 2 = 5x$$

$$\Rightarrow x = \frac{1}{5} (\sin x + 2)$$

$$\Rightarrow g(x) = \frac{1}{5} (\sin x + 2)$$

$$g'(x) = \frac{1}{5} \cos x$$

$$|g'(x)| < 1 \quad \because |\cos x| < 1$$

So it is converges so setting iteration is

$$x_{n+1} = g(x_n)$$

$$\Rightarrow x_{n+1} = \frac{1}{5} (\sin x_n + 2) \rightarrow (i)$$

Let guess root is 0.4 be initial guess.

1<sup>st</sup> Iteration:

Put  $n=0$  in (i)

$$\begin{aligned} \Rightarrow x_1 &= \frac{1}{5} (2 + \sin x_0) \\ &= \frac{1}{5} (2 + \sin(0.4)) \\ &= 0.4779 \end{aligned}$$

2<sup>nd</sup> Iteration:

$$\begin{aligned} \text{Put } n=1 \text{ in (i)} \\ x_2 &= \frac{1}{5} (2 + \sin x_1) \\ &= \frac{1}{5} (2 + \sin(0.4779)) \\ &= 0.4919 \end{aligned}$$

3<sup>rd</sup> Iteration:

$$\begin{aligned} \text{Put } n=2 \text{ in (i)} \\ x_3 &= \frac{1}{5} (2 + \sin x_2) \\ &= \frac{1}{5} (2 + \sin(0.4919)) \\ &= 0.4945 \end{aligned}$$

4<sup>th</sup> Iteration:

$$\begin{aligned} \text{Put } n=3 \text{ in (i)} \\ x_4 &= \frac{1}{5} (2 + \sin x_3) \\ &= \frac{1}{5} (2 + \sin(0.4945)) \\ &= 0.4949 \end{aligned}$$

5<sup>th</sup> Iteration:

$$\begin{aligned} \text{Put } n=4 \text{ in (i)} \\ x_5 &= \frac{1}{5} (2 + \sin x_4) \\ &= \frac{1}{5} (2 + \sin(0.4949)) \\ &= 0.4949 \end{aligned}$$

So  $x = 0.4949$  is required root.

(Answer).

Example:-

Find a real root correct to 4 d.p of equation  $x^3 + x - 1 = 0$  by iteration Method.

Solution:- Let  $f(x) = x^3 + x - 1$

|        |    |   |
|--------|----|---|
| $x$    | 0  | 1 |
| $f(x)$ | -1 | 1 |

As  $f(0) = -1 < 0$

$f(1) = 1 > 0$

∴ root lies between 0 and 1.

Now  $x^3 + x - 1 = 0$

$x^2(x+1) = 1$

$x^2 = \frac{1}{1+x}$

$x = \frac{1}{\sqrt{1+x}} \rightarrow (i)$

$g(x) = (1+x)^{-1/2}$

$g'(x) = \frac{1}{2} (1+x)^{-3/2}$

$= \frac{1}{2(1+x)^{3/2}}$

$|g'(x)| = \left| \frac{1}{2(1+x)^{3/2}} \right| < 1$   
for  $x \in [0, 1]$

∴ Iteration method is applicable.

Setting iteration to (i),

$x_{n+1} = \frac{1}{\sqrt{1+x_n}} \rightarrow (*)$

Now Guess root taken as

$x_0 = 1$

1<sup>st</sup> Iteration:

Put  $n=0$  in (\*),

$x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+1}}$

$x_1 = 0.7071$

2<sup>nd</sup> Iteration:

Put  $n=1$  in (\*),

$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.7071}}$

$= 0.7654$

3<sup>rd</sup> Iteration:

Put  $n=2$  in (\*),

$x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.7654}}$

$= 0.7526$

4<sup>th</sup> Iteration:

Put  $n=3$  in (\*),

$x_4 = \frac{1}{\sqrt{1+x_3}}$

$= \frac{1}{\sqrt{1+0.7526}}$

$= 0.7554$

5<sup>th</sup> Iteration:

Put  $n=4$  in (\*),

$x_5 = \frac{1}{\sqrt{1+x_4}}$

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$$= \frac{1}{\sqrt{0.7554+1}}$$

$$= 0.7548$$

6<sup>th</sup> Iteration:Put  $n=5$  in (\*)

$$x_6 = \frac{1}{\sqrt{1+x_5}}$$

$$= \frac{1}{\sqrt{0.7548+1}}$$

$$= 0.7549$$

7<sup>th</sup> Iteration:Put  $n=6$  in (\*)

$$x_7 = \frac{1}{\sqrt{1+x_6}}$$

$$= \frac{1}{\sqrt{0.7549+1}}$$

$$= 0.7549$$

Required root is 0.7549.

(Answer)

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EXERCISE

- 1- Find the root of equation  $2x - \cos x - 3 = 0$  upto 3 decimal places by simple iteration Method.

Ans:  $1.5235 \approx 1.524$ 

- 2- Solve  $x + x^x = 100$  by iteration method upto 4 d.p.

Ans: 3.5812

- 3- Find the real root of equation  $x - \sin x = 0.25$  using fixed point iteration

Ans: 1.1712

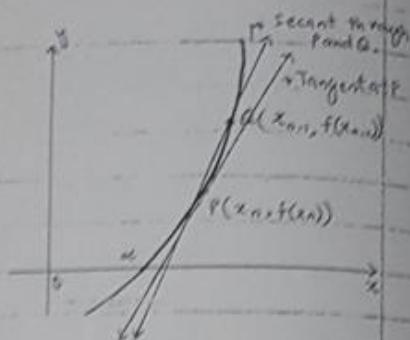
- 4- Set iterative formula for solving the equation  $x^{n+1} - \cos x - 10 = 0$ , as starting with  $x = 1.5$

Ans: 1.4611



## 5- SECANT METHOD:-

If  $x_{n+1}$  and  $x_n$  are the two approximation in the root  $\alpha$  of  $f(x) = 0$ , then in better approximation  $x_{n+1}$  to the root  $\alpha$  is given by



$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Proof:-

Let  $P(x_n, f(x_n))$  and  $Q(x_{n+1}, f(x_{n+1}))$  be two point on the curve. Then slope of tangent at P  $\approx$  Slope of Secant through P and Q.

$$\Rightarrow f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \rightarrow (i)$$

Also by Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow (ii)$$

Put value  $f'(x_n)$  in (ii)

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

which is required Secant formula.

Example:-

Find root of  $x \log_{10} x - 1.2 = 0$  upto 4 decimal places by Secant Method.

Solution:-

Let  $f(x) = x \log_{10} x - 1.2$

| $x$    | 1    | 2       | 3      |
|--------|------|---------|--------|
| $f(x)$ | -1.2 | -0.5979 | 0.2314 |

$\Rightarrow$  root lies between 2 and 3.

Let  $x_0 = 2$ ,  $x_1 = 3$

$f(x_0) = -0.5979$ ,  $f(x_1) = 0.2314$

Now by Secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \rightarrow (i)$$

I<sup>st</sup> Iteration:

Put  $n = 1$  in (i)

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)}$$

$$\begin{aligned}
 x_2 &= 2.7210 \\
 f(x_2) &= f(2.7210) \\
 &= (2.7210) \log_{10} (2.7210) - 1.2 \\
 &= -0.0171
 \end{aligned}$$

2<sup>nd</sup> Iteration:

Put  $n=2$  in (i)

$$\begin{aligned}
 x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\
 &= \frac{3(-0.0171) - (2.7210)(0.2314)}{-0.0171 - 0.2314} \\
 &= 2.7404
 \end{aligned}$$

$$\begin{aligned}
 f(x_3) &= f(2.7404) \\
 &= (2.7404) \log_{10} (2.7404) - 1.2 \\
 &= -0.0002
 \end{aligned}$$

3<sup>rd</sup> Iteration:

Put  $n=3$  in (i)

$$\begin{aligned}
 x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\
 &= \frac{(2.7210)(-0.0002) - (2.7404)(-0.0171)}{-0.0002 - (-0.0171)} \\
 &= 2.7406
 \end{aligned}$$

$$\begin{aligned}
 f(x_4) &= f(2.7406) \\
 &= (2.7406) \log_{10} (2.7406) - 1.2 \\
 &= 0.0000
 \end{aligned}$$

So required root is 2.7406

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Example:-

Find positive root of equation  $x^3 - x - 10 = 0$  by Secant method.

Solution:- Let  $f(x) = x^3 - x - 10$

|        |     |        |     |        |    |       |    |
|--------|-----|--------|-----|--------|----|-------|----|
| $x$    | 0   | 0.5    | 1   | 1.5    | 2  | 2.5   | 3  |
| $f(x)$ | -10 | -10.37 | -10 | -8.125 | -4 | 3.125 | 14 |

$\Rightarrow$  root lies between 2 and 2.5.

$$\begin{aligned}
 x_0 &= 2 & x_1 &= 2.5 \\
 f(x_0) &= -4 & f(x_1) &= 3.125
 \end{aligned}$$

Now by Secant Method

$$x_{n+1} = \frac{x_n f(x_n) - x_{n-1} f(x_{n-1})}{f(x_n) - f(x_{n-1})} \rightarrow (i)$$

1<sup>st</sup> Iteration:

Put  $n=1$  in (i)

$$\begin{aligned}
 x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\
 &= \frac{2(3.125) - 2.5(-4)}{3.125 - (-4)} \\
 &= 2.281
 \end{aligned}$$

$$\begin{aligned}
 f(x_2) &= f(2.281) \\
 &= (2.281)^3 - (2.281) - 10 \\
 &= -0.413
 \end{aligned}$$

2<sup>nd</sup> Iteration:

Put  $n=2$  in (i)

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

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$$= \frac{2.5(-0.413) - (2.281)(3.125)}{-0.413 - 3.125}$$

$$x_3 = 2.307$$

$$f(x_3) = f(2.307)$$

$$= (2.307)^3 - (2.307) - 10$$

$$= -0.029$$

3<sup>rd</sup> Iteration:Put  $n=3$  in (i),

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{2.281(-0.029) - (2.307)(-0.413)}{-0.029 - (-0.413)}$$

$$= 2.309$$

$$f(x_4) = f(2.309)$$

$$= (2.309)^3 - (2.309) - 10$$

$$= (2.309)^3 - 12.309$$

$$= 0.001$$

So approximation root is 2.309.

(Answer)

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EXERCISE

- 1- By taking logarithm, find the solution of  $x^x = 10$  correct to 4 significant figure by Secant method.

Ans: 2.5060

- 2- Using Secant method calculate a root of  $x^3 + 2x^2 + 10x - 20 = 0$ .

Ans: 1.3688

- 3- Solve equation  $\sin x - 5x + 2 = 0$  by Secant method correct to 3 dp given root lies between 0.6 and 0.4.

Ans: 0.4950

- 4- Use Secant formula to compute a real root of equation  $x^4 - x - 10 = 0$

- 5- Use secant Method find root of  $\sin x = \frac{x}{2}$ .

Ans: 0

- 6- Find smallest positive root of equation  $f(x) = \cos x \cosh x - 1$  by secant method.

Ans: 4.730



## \* ORDER OF CONVERGENCE:-

If  $e_n$  denotes the absolute error in  $n^{\text{th}}$  iteration and  $e_{n+1}$  denoted the absolute error in  $(n+1)^{\text{th}}$  iteration, then  $P$  is said to be order of convergence if

$$e_{n+1} \propto e_n^p$$

i.e.  $e_{n+1} = k e_n^p$  for some  $k$ .

## \* Order of Convergence of Fixed point iterative Method

Prove that order of convergence of fixed point iteration method is one.

(OR)

Fixed point iteration method is linearly convergent.

Solution:

As given equation  $f(x) = 0 \rightarrow (i)$   
by iteration method we write it as

$$x = g(x) \rightarrow (ii)$$

Setting iteration as

$$x_{n+1} = g(x_n) \rightarrow (iii)$$

Let  $x = a$  be root of equation (i)

$$f(a) = 0$$

Also from (ii)

$$a = g(a) \rightarrow (iv)$$

Let  $e_n$  be error in  $n^{\text{th}}$  term and  $e_{n+1}$  be the error in  $(n+1)^{\text{th}}$  term.

$$x_n - a = e_n$$

$$x_{n+1} - a = e_{n+1}$$

$$\Rightarrow x_n = e_n + a, \quad x_{n+1} = e_{n+1} + a$$

Put in equation (iii)

$$e_{n+1} + a = g(a + e_n)$$

By Taylor's Series

$$a + e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \dots$$

$$= a + e_n g'(a)$$

$$\because g(a) = a$$

$\therefore$  Neglecting higher power of  $e_n$

$$a + e_{n+1} = a + e_n g'(a)$$

$$\Rightarrow e_{n+1} = e_n g'(a)$$

$$\Rightarrow e_{n+1} = g'(a) e_n^{(1)}$$

$$e_{n+1} = k e_n^{(1)} \quad \text{where } g'(a) = k$$

Which shows that order of convergence of fixed point iteration method is one. (OR)

Iteration method is linearly convergent.

(proved)

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THEOREM:-

Suppose that if equation  $x = g(x)$  has a solution and continuous in  $[a, b]$ . If  $|g'(x)| < k < 1$  for  $x \in [a, b]$ , then show that iteration  $x_{i+1} = g(x_i)$  will converge for any starting point in  $[a, b]$ .

Proof:- Let  $f(x) = 0$  be a given equation and we also write it as  $x = g(x) \rightarrow (i)$

Setting the iteration

$$x_{n+1} = g(x_n) \rightarrow (ii)$$

Let  $x = a$  be the root of equation i.e.  $f(a) = 0$

Also from (i)

$$a = g(a) \rightarrow (iii)$$

Let  $e_n$  and  $e_{n+1}$  be the error terms in  $n^{th}$  and  $(n+1)^{th}$  iteration respectively

$$x_n - a = e_n$$

$$x_{n+1} - a = e_{n+1}$$

$$\Rightarrow x_n = a + e_n$$

$$\text{and } x_{n+1} = a + e_{n+1}$$

put in (ii) we get

$$a + e_{n+1} = g(a + e_n)$$

By Taylor's Series

$$a + e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \dots$$

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$$a + e_{n+1} = g(a) + e_n g'(a)$$

= By Neglecting square and higher power of  $e_n$ .

$$g(a) + e_{n+1} = a + e_n g'(a) \quad (\because a = g(a))$$

$$e_{n+1} = e_n g'(a)$$

$$|e_{n+1}| = |e_n| |g'(a)|$$

$$\frac{|e_{n+1}|}{|e_n|} = |g'(a)|$$

$$\therefore \frac{|e_{n+1}|}{|e_n|}$$

$$\text{If } |g'(a)| < 1$$

$$\frac{|e_{n+1}|}{|e_n|} < 1$$

$$|e_{n+1}|$$

$$\Rightarrow |e_{n+1}| < |e_n|$$

The magnitude of error decreases in each step and the fixed point iteration method converges.

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## \* ORDER OF CONVERGENCE

(Newton Raphson Method).

### Theorem:

Prove that order of Convergence of Newton-Raphson method is 2.  
(OR)

Prove that Newton-Raphson method is quadratically Convergent.

### Proof:-

Let  $x=a$  be root of the equation  $f(x)=0$  and then  $f(a)=0$ .

Now;

If we denote  $n^{th}$  approximation to root by  $x_n$  and error in  $n^{th}$  approx. to the root by  $e_n$ , then

$$e_n = x_n - a$$

$$\text{and } e_{n+1} = x_{n+1} - a$$

$$\Rightarrow x_n = e_n + a$$

$$x_{n+1} = e_{n+1} + a$$

Now by Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$e_{n+1} + a = e_n + a - \frac{f(a+e_n)}{f'(a+e_n)}$$

$$e_{n+1} = e_n - \frac{f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \dots}$$

$$e_{n+1} = e_n - \frac{e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \dots}$$

$\therefore f(a) = 0$

Neglecting square and higher power of  $e_n$ , we have

$$e_{n+1} = e_n - \frac{e_n f'(a)}{f'(a) + e_n f''(a)}$$

$$= e_n - \frac{e_n f'(a)}{f'(a) \left[ 1 + \frac{e_n f''(a)}{f'(a)} \right]}$$

$$= e_n - e_n \left[ 1 + \frac{e_n f''(a)}{f'(a)} \right]^{-1}$$

$$= e_n - e_n \left[ 1 - \frac{e_n f''(a)}{f'(a)} \right]$$

$\therefore$  Neglecting remaining term

$$= e_n - e_n + \frac{e_n^2 f''(a)}{f'(a)}$$

$$e_{n+1} = \frac{e_n^2 f''(a)}{f'(a)}$$

$$e_{n+1} = k e_n^2, \quad k = \frac{f''(a)}{f'(a)}$$

$\Rightarrow$  Newton Raphson method has order of Convergence 2. i.e. Newton Raphson method is quadratically Convergent.



\* Theorem:-

Prove that Newton-Raphson method is linearly Convergent if there exists multiple roots.

Proof:-

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow (i)$$

Let  $x=a$  be root of  $f(x)=0$ , then  $f(a)=0$ . Now with usual meanings

$$e_n = x_n - a, \quad e_{n+1} = x_{n+1} - a$$

$$\Rightarrow x_n = e_n + a, \quad x_{n+1} = e_{n+1} + a$$

then (i)  $\Rightarrow$

$$e_{n+1} + a = e_n + a - \frac{f(e_n + a)}{f'(e_n + a)}$$

$$e_{n+1} = e_n - \frac{f(a + e_n)}{f'(a + e_n)}$$

$$= e_n - \frac{f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \dots}$$

Now if  $x=a$  be a double root then  $f(a)=0$  also  $f'(a)=0$  then

$$e_{n+1} = e_n - \frac{e_n^2/2! f''(a) + \dots}{e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \dots}$$

$$= e_n - e_n \left[ \frac{e_n/2! f''(a) + \dots}{f''(a) + \frac{e_n}{2!} f'''(a) + \dots} \right]$$

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$$= e_n - \frac{e_n}{2} \left[ \frac{f''(a) + \dots}{f''(a) + \frac{e_n}{2} f'''(a) + \dots} \right]$$

$$= \frac{e_n^2}{2} - \frac{e_n}{2} \left[ \frac{f''(a)}{f''(a)} \right] \quad \text{Neglecting remaining terms}$$

$$= e_n - \frac{e_n}{2}$$

$$e_{n+1} = \frac{1}{2} e_n \quad (1)$$

Continuing in this way if  $x=a$  is a triple root then

$$e_{n+1} = \frac{1}{3} e_n$$

Also for multiplicity root  $n$  is given

$$e_{n+1} = \frac{1}{n} e_n$$

Which shows that order of convergence of Newton-Raphson for multiple roots is Linear.

(proved).

\* Question:-

Show that Newton-Raphson method converges if  $|f(x)| < |f'(x)|^2$   
 $|f''(x)|$

Solution:-

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow (i)$$

By Fixed point iteration method

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$$x_{n+1} = g(x_n) \rightarrow (ii)$$

From (i) and (ii)

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow g'(x) = 1 - \frac{[f'(x)f'(x) - f(x)f''(x)]}{(f'(x))^2}$$

$$= \frac{(f'(x))^2 - (f'(x))^2 + f(x)f''(x)}{(f'(x))^2}$$

$$|g'(x)| = \left| \frac{f(x)f''(x)}{(f'(x))^2} \right|$$

For Convergence  $|g'(x)| < 1$

$$\Rightarrow \frac{f(x)f''(x)}{(f'(x))^2} < 1$$

$$\Rightarrow \frac{|f(x)|}{|f''(x)|} < \frac{|f'(x)|^2}{|f''(x)|}$$

(proved)

### \* Advantages of Newton-Raphson Method :-

- (i) It is fast convergent iterative method its convergence is quadratic.
- (ii) Newton-Raphson method can be used to calculate the complex root of non-linear equation.
- (iii) Newton's Method is useful in case of large value of  $f'(x)$ .  
i.e. When graph of function is  $f(x)$  while crossing  $x$ -axis nearly vertical.
- (iv) Newton's method is generally used to improve the result obtain by other methods.
- (v) It is applicable to solution of both algebraic and transcendental function.

### \* Disadvantages of Newton-Raphson Method :-

- (i) If two or more roots are equal or nearly equal, the method is not fastly convergent. In this case the convergence is Linear.
- (ii) If root is very near to maximum or minimum value of function  $f'(x) = 0$  at such point Newton's method fails.



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- (iii) The evaluation of derivative  $f'(x)$  at each step makes the method unpopular computationally and thus is not well-adopted to computer usage.

### Remark:-

The most important characteristics of order of convergence is that, that if an iteration method has large order of convergence then it will fastly converges to the root. From this we conclude that Newton-Raphson Method most rapidly converges to the root.

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### \* ORDER OF CONVERGENCE

(Secant Method)

### Question:-

Evaluate the rate of convergence of Secant Method (Error Analysis)?

### Solution:-

By Secant Method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \quad \text{--- (i)}$$

Let 'a' be the exact root of the equation  $f(x) = 0$ , then  $f(a) = 0$ .

Also Let  $x_{n-1} - a = e_{n-1}$

$$x_n - a = e_n$$

$$\Rightarrow x_{n-1} = a + e_{n-1}$$

$$x_n = a + e_n$$

$$\text{and } x_{n+1} = a + e_{n+1}$$

Using all these values in (i) we get-

$$a + e_{n+1} = a + e_n - \frac{(a + e_n - a + e_{n-1}) f(a + e_n)}{f(a + e_n) - f(a + e_{n-1})}$$

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}] [f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots]}{[f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots] - [f(a) + e_{n-1} f'(a) + \frac{e_{n-1}^2}{2!} f''(a) + \dots]}$$

As  $f(a) = 0$  we get,

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$$\begin{aligned}
 e_{n+1} - e_n &= \frac{(e_n - e_{n-1}) e_n f''(a)}{(e_n - e_{n-1}) f'(a) + \left(\frac{e_n^2 - e_{n-1}^2}{2}\right) f''(a) + \dots} \\
 &= \frac{e_n - e_n}{\left[1 + \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} + \dots\right]} \\
 &= \frac{e_n - e_n}{\left[1 + \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} + \dots\right]} \left[1 + \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} + \dots\right]^{-1} \\
 &= \frac{e_n - e_n}{\left[1 + \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} + \dots\right]} \left[1 - \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} + \dots\right] \\
 &\approx \text{Neglecting remaining terms.}
 \end{aligned}$$

$$\begin{aligned}
 &= e_n - e_n \left[1 + \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} - \frac{e_n + e_{n-1}}{2} \frac{f''(a)}{f'(a)} - \frac{e_n^2 f''(a)}{2 f'(a)} + \frac{e_n(e_n + e_{n-1}) f''(a)}{2 f'(a)} + \frac{e_n^2(e_n + e_{n-1})}{4} \left(\frac{f''(a)}{f'(a)}\right)^2\right] \\
 &= \frac{-e_n^2 f''(a)}{2 f'(a)} + \frac{e_n^2 f''(a)}{2 f'(a)} + \frac{e_n e_{n-1} f''(a)}{2 f'(a)} \\
 &(\because \text{Neglecting remaining terms})
 \end{aligned}$$

$$\Rightarrow e_{n+1} = \frac{1}{2} \frac{f''(a)}{f'(a)} \cdot e_n e_{n-1}$$

$$\Rightarrow e_{n+1} = k \cdot e_n e_{n-1}$$

$$\text{where } k = \frac{1}{2} \frac{f''(a)}{f'(a)}$$

$$\Rightarrow e_{n+1} \propto e_n e_{n-1}$$

This shows that in every iteration error is proportional to the product of the error in two previous iteration. Hence Secant method Converges.

### \* Theorem:-

Prove that order of Convergence of Secant Method is approximately 1.618.

### Proof:-

Let  $P$  be the order of Convergence of Secant Method, then with the usual meanings

$$e_{n+1} = k e_n^P \rightarrow (i)$$

Now also as secant method is  $(e_{n+1})$  proportional to  $e_n e_{n-1}$

$$\text{i.e. } e_{n+1} \propto e_n \cdot e_{n-1}$$

$$\text{So let } e_{n+1} = k_1 e_n e_{n-1} \rightarrow (ii)$$

Now (i) can be written as

$$e_n = k e_{n-1}^P$$

$$\Rightarrow e_n^{1/P} = k^{1/P} e_{n-1}$$

$$\Rightarrow e_{n-1} = \frac{e_n^{1/P}}{k^{1/P}}$$

Put in (ii)

$$e_{n+1} = k_1 e_n \cdot \frac{e_n^{1/P}}{k^{1/P}}$$

$$\Rightarrow k e_n^P = \frac{k_1 e_n^{1+1/P}}{k^{1/P}}$$

( $\because$  By (i))

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Comparing powers of  $e^n$ , we get

$$P = \frac{1}{P} + 1$$

$$P^2 = 1 + P$$

$$\Rightarrow P^2 - P - 1 = 0$$

$$P = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

As  $P$  is positive

$$\Rightarrow P = \frac{1 + \sqrt{5}}{2}$$

$$P = 1.618$$

Hence order of convergence of Secant method is 1.6180.

Remark:-

In Secant formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

If  $f(x_n) = f(x_{n-1})$ , Secant method fails.

This shows that it does not converge always. This is a drawback of Secant method over the method of false position which always converges, But if Secant method converges its rate of convergence is faster than the method of false position.

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\* **LU**

Decomposition (OR)  
Triangular Factorization Method (OR)  
Triangular Decomposition Method.

Given system of equation  
 $AX = b \rightarrow (i)$

Decompose (or factorize)  $A$  into two square matrices  $L$  and  $U$ , where  $L$  is lower triangular matrix and  $U$  is upper triangular matrix, i.e.  $A = LU$

Equation (i) becomes

$$(LU)X = b$$

$$L(UX) = b \rightarrow (ii)$$

$$\text{Let } UX = y$$

$$\text{then (ii) becomes } Ly = b$$

Working Rule:-

(i) Express  $A = LU$

(ii) From  $Ly = b$  Find  $y$

(iii) From  $UX = y$  Find  $x$

In the decomposition  $A = LU$ ,  $L$  and  $U$  are not unique. To find a unique decomposition, the diagonal entries of  $L$  or  $U$  are specified by

- i) Doolittle's Method
- ii) Crout's Method
- iii) Cholesky's Method.

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# 1) DOLITTLE'S METHOD:-

In Dolittle's method, all diagonal elements of  $L$  are restricted to be one.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

For inverse of matrix by Dolittle's method is

(i)  $A = LU$

(ii)  $A^{-1} = (LU)^{-1}$

$$A^{-1} = U^{-1}L^{-1}$$

$$UA^{-1} = (UU^{-1})L^{-1}$$

$$UA^{-1} = L^{-1}$$

(iii)  $LL^{-1} = I$

$$L^{-1} = ?$$

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Example:-

Solve the system of equation

$$2x_1 + x_2 + 3x_3 = 11$$

$$4x_1 + 3x_2 + 10x_3 = 28$$

$$2x_1 + 4x_2 + 17x_3 = 31$$

By Dolittle's method. Also find  $A^{-1}$ .

Solutions:-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$A \quad \quad \quad x = b$

Let  $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements of matrix we get

$$\Rightarrow u_{11} = 2$$

$$\Rightarrow u_{12} = 1$$

$$\Rightarrow u_{13} = 3$$

$$l_{21}u_{11} = 4$$

$$\Rightarrow l_{21} = 2$$

$$l_{31}u_{11} = 2$$

$$\Rightarrow l_{31} = 1$$

$$l_{21}u_{12} + u_{22} = 3$$

$$2(1) + u_{22} = 3$$

$$u_{22} = 3 - 2$$

$$\Rightarrow u_{22} = 1$$

$$l_{21}u_{13} + u_{23} = 10$$

$$(2)(3) + u_{23} = 10$$

$$\Rightarrow u_{23} = 4$$

$$l_{31}u_{12} + l_{32}u_{22} = 4$$

$$(1)(1) + l_{32}(1) = 4$$

$$l_{32} = 4 - 1$$

$$\Rightarrow l_{32} = 3$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 17$$

$$(1)(3) + (3)(4) + u_{33} = 17$$

$$u_{33} = 17 - 12 - 3$$

$$\Rightarrow u_{33} = 2$$

we have

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$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Let  $LY = b$ 

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ y_1 + 3y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\Rightarrow y_1 = 11$$

$$2y_1 + y_2 = 28$$

$$2(11) + y_2 = 28$$

$$\Rightarrow y_2 = 6$$

$$\text{and } y_1 + 3y_2 + y_3 = 31$$

$$(11) + 3(6) + y_3 = 31$$

$$\Rightarrow y_3 = 31 - 11 - 18$$

$$y_3 = 2$$

$$\therefore Y = (11, 6, 2)^T$$

$$\text{Now, } UX = Y$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ 2 \end{bmatrix}$$

$$2x_1 + x_2 + 3x_3 = 11 \rightarrow (i)$$

$$x_2 + 4x_3 = 6 \rightarrow (ii)$$

$$2x_3 = 2 \rightarrow (iii)$$

From above equation

$$\Rightarrow x_3 = 1$$

$$\Rightarrow x_2 = 6 - 4(1) = 2$$

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$$2x_1 + 2 + 3(1) = 11$$

$$2x_1 = 11 - 3 - 2$$

$$x_1 = \frac{6}{2} = 3$$

$$\Rightarrow x_1 = 3, x_2 = 2, x_3 = 1$$

So solution vector is  $X = (3, 2, 1)^T$ Also for  $A^{-1}$ :First we find  $L^{-1}$ 

$$\text{As } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$LL^{-1} = L^{-1}L = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2+l_{21}' & 1 & 0 \\ 1+l_{31}'+l_{32}' & 3+l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing corresponding elements we have

$$2 + l_{21}' = 0, \quad 3 + l_{32}' = 0$$

$$\Rightarrow l_{21}' = -2$$

$$\Rightarrow l_{32}' = -3$$

and

$$1 + 3l_{31}' + l_{32}' = 0$$

$$1 + 3(-2) + l_{31}' = 0$$

$$l_{31}' = 6 - 1$$

$$\Rightarrow l_{31}' = 5$$

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$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

Now,

$$UA^{-1} = L^{-1}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} + a_{21} + 3a_{31} & 2a_{12} + a_{22} + 3a_{32} & 2a_{13} + a_{23} + 3a_{33} \\ a_{21} + 4a_{31} & a_{22} + 4a_{32} & a_{23} + 4a_{33} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$2a_{31} = 5$$

$$\Rightarrow a_{31} = 5/2$$

$$2a_{32} = -3$$

$$\Rightarrow a_{32} = -3/2$$

$$a_{22} + 4a_{32} = 1$$

$$a_{22} = 1 - 4(-3/2)$$

$$\Rightarrow a_{22} = 7$$

$$2a_{11} + a_{21} + 3a_{31} = 1$$

$$2a_{11} - 3 \times 4 + 3 \left( \frac{5}{2} \right) = 1$$

$$a_{11} = \frac{11}{4}$$

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$$2a_{11} + a_{22} + 3a_{33} = 0$$

$$2(a_{11}) + 7 + 3 \left( \frac{-1}{2} \right) = 0$$

$$\Rightarrow a_{11} = \frac{-1}{2}$$

$$2a_{12} + a_{22} + 3a_{33} = 0$$

$$3(a_{12}) + (-3) + 3 \left( \frac{1}{2} \right) = 0$$

$$\Rightarrow a_{12} = \frac{1}{4}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 11/4 & -1/4 & 3/4 \\ -5/2 & 7 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

(Answer)

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$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

Now;

$$UA^{-1} = L^{-1}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} + a_{21} + 3a_{31} & 2a_{12} + a_{22} + 3a_{32} & 2a_{13} + a_{23} + 3a_{33} \\ a_{21} + 4a_{31} & a_{22} + 4a_{32} & a_{23} + 4a_{33} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$2a_{31} = 5$$

$$\Rightarrow a_{31} = 5/2$$

$$2a_{32} = -3$$

$$\Rightarrow a_{32} = -3/2$$

$$a_{22} + 4a_{32} = 1$$

$$a_{22} = 1 - 4(-3/2)$$

$$\Rightarrow a_{22} = 7$$

$$2a_{11} + a_{21} + 3a_{31} = 1$$

$$2a_{11} - 3 \times 5 + 3(5/2) = 1$$

$$a_{11} = \frac{11}{4}$$

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$$2a_{12} + a_{22} + 3a_{32} = 0 \quad , \quad 3a_{13} + a_{23} + 3a_{33} = 0$$

$$2(a_{12}) + 7 + 3(-3/2) = 0 \quad , \quad 3(a_{13}) + (-2) + 3(1/2) = 0$$

$$\Rightarrow a_{12} = -1/3 \quad , \quad \Rightarrow a_{13} = 1/4$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

(Answer).

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2) ROUT'S METHOD:-

In Crout's method, the upper triangular matrix  $U$  has unit diagonal elements.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

For inverse of matrix by Crout's Method:-

(i)  $A = LU$

(ii)  $A^{-1} = (LU)^{-1}$

$\Rightarrow A^{-1} = U^{-1}L^{-1}$

$A^{-1}L = U^{-1}(L^{-1}L)$

$A^{-1}L = U^{-1}$

(iii)  $UU^{-1} = U^{-1}U = I$

Find  $U^{-1}$ .

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Example:-

Solve following system of equations

$$2x_1 + x_2 + 3x_3 = 11$$

$$4x_1 + 3x_2 + 10x_3 = 28$$

$$2x_1 + 4x_2 + 17x_3 = 31$$

by Crout's method Also find  $A^{-1}$  by using Crout's method.

Solution:-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \underline{x} = b$$

Let  $LU = A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements of matrix.

$\Rightarrow l_{11} = 2$

$\Rightarrow l_{21} = 4$

$\Rightarrow l_{31} = 2$

$l_{11}u_{12} = 1$

$\Rightarrow u_{12} = \frac{1}{2}$

$l_{11}u_{13} = 3$

$\Rightarrow u_{13} = \frac{3}{2}$

$l_{21}u_{12} + l_{22} = 3$

$(4)(\frac{1}{2}) + l_{22} = 3$

$\Rightarrow l_{22} = 1$

$l_{21}u_{13} + l_{22}u_{23} = 10$

$(4)(\frac{3}{2}) + 1u_{23} = 10$

$\Rightarrow u_{23} = 4$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 17$

$(2)(\frac{3}{2}) + 3(4) + l_{33} = 17$

$l_{33} = 17 - 12 - 3$

$\Rightarrow l_{33} = 2$

So

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\text{Let } LY = b$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} 2y_1 \\ 4y_1 + y_2 \\ 2y_1 + 3y_2 + 2y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\Rightarrow y_1 = 11/2$$

$$4y_1 + y_2 = 28$$

$$\Rightarrow y_2 = 28 - 4\left(\frac{11}{2}\right) = 6$$

$$2y_1 + 3y_2 + 2y_3 = 31$$

$$2\left(\frac{11}{2}\right) + 3(6) + 2y_3 = 31$$

$$\Rightarrow y_3 = 1$$

$$Y = \left[ \frac{11}{2}, 6, 1 \right]^T$$

$$\text{Now } UX = Y$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 6 \\ 1 \end{bmatrix}$$

$$x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_3 = \frac{11}{2} \rightarrow (i)$$

$$x_2 + 4x_3 = 6 \rightarrow (ii)$$

$$x_3 = 1 \rightarrow (iii)$$

Solving above equation

$$\Rightarrow x_3 = 1$$

$$\Rightarrow x_2 = 6 - 4(1) = 2$$

$$\Rightarrow x_1 = \frac{11}{2} - 1 - \frac{3}{2} = \frac{11}{2} - 1 - \frac{3}{2}$$

$$= \frac{11}{2} - 1 - \frac{3}{2}$$

$$x_1 = 1 + 2 = 3$$

$$\text{So Solution Vector } X = [3, 2, 1]^T$$

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Now Also for  $A^{-1}$ :

First we calculate  $U^{-1}$ .

$$\text{As } UU^{-1} = U^{-1}U = I$$

$$\begin{bmatrix} 1 & u_{12}' & u_{13}' \\ 0 & 1 & u_{23}' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 + u_{12}' & 3/4 + u_{13}' \\ 0 & 1 & 4 + u_{23}' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow u_{12}' = -1/2$$

$$\Rightarrow u_{13}' = -3/4 - 4\left(-\frac{1}{2}\right) = \frac{5}{4}$$

$$\Rightarrow u_{23}' = -4$$

So

$$U^{-1} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$$A^{-1}L = U^{-1}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} + 4a_{12} + 2a_{13} & a_{12} + 3a_{13} & 2a_{13} \\ 2a_{21} + 4a_{22} + 2a_{23} & a_{22} + 3a_{23} & 2a_{23} \\ 2a_{31} + 4a_{32} + 2a_{33} & a_{32} + 3a_{33} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a_{13} = 1/4, \quad a_{12} + 3a_{13} = -1/2 \rightarrow (iv)$$

$$\Rightarrow a_{23} = -2, \quad a_{22} + 3a_{23} = 1 \rightarrow (v)$$

$$\Rightarrow a_{33} = 1/2, \quad a_{32} + 3a_{33} = 0 \rightarrow (vi)$$

$$2a_{11} + 4a_{12} + 2a_{13} = 1 \rightarrow (vi)$$

$$2a_{21} + 4a_{22} + 2a_{23} = 0 \rightarrow (vii)$$

$$2a_{31} + 4a_{32} + 2a_{33} = 0 \rightarrow (ix)$$

From above equations we get

$$a_{11} = 1/4$$

$$a_{22} = -5/4$$

$$a_{21} = -12$$

$$a_{22} = 7$$

$$a_{31} = 5/2$$

$$a_{32} = -3/2$$

$$\text{So } A^{-1} = \begin{bmatrix} 1/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

(Answer)

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### 3) CHOLESKY'S METHOD:-

In Cholesky's method, the diagonal elements of  $L$  and  $U$  are same, then for this method

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Note: If  $A$  is symmetric then

$$A = LL^T \quad \therefore U = L^T$$

$$A^{-1} = (L^{-1})^T L^{-1}$$

Example:-

Solve by Cholesky's Method

$$2x_1 + 1x_2 + 3x_3 = 11$$

$$4x_1 + 3x_2 + 10x_3 = 28$$

$$2x_1 + 4x_2 + 17x_3 = 31$$

Solution:-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \cdot x = b$$

$$\text{Let } A = LU$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$



$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}^2 & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}l_{11} & l_{21}u_{12}+l_{22}^2 & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}l_{11} & l_{31}u_{12}+l_{32}l_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing elements of matrices we get

$$l_{11}^2 = 2$$

$$\Rightarrow l_{11} = \sqrt{2}$$

$$l_{11}u_{12} = 1$$

$$\Rightarrow u_{12} = \frac{1}{\sqrt{2}}$$

$$l_{21}l_{11} = 4$$

$$\Rightarrow l_{21} = \frac{4}{\sqrt{2}}$$

$$l_{31}l_{11} = 2$$

$$\Rightarrow l_{31} = \frac{2}{\sqrt{2}}$$

$$l_{21}u_{12} + l_{22}^2 = 3$$

$$\frac{4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + l_{22}^2 = 3$$

$$\Rightarrow l_{22}^2 = 3 - 2 = 1$$

$$\Rightarrow l_{22} = 1$$

$$l_{31}u_{12} + l_{32}l_{22} = 4$$

$$\frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + l_{32} = 4$$

$$\Rightarrow l_{32} = 4 - 1 = 3$$

$$\Rightarrow l_{32} = 3$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 = 17$$

$$\frac{2}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}} + 3(4) + l_{33}^2 = 17$$

$$3 + 12 + l_{33}^2 = 17$$

$$l_{33}^2 = 17 - 15 = 2$$

$$\Rightarrow l_{33} = \sqrt{2}$$

$$\Rightarrow l_{33} = \sqrt{2}$$

$$\Rightarrow l_{33} = \sqrt{2}$$

$$\Rightarrow l_{33} = \sqrt{2}$$

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put in L and U we get

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{4}{\sqrt{2}} & 1 & 0 \\ \sqrt{2} & 3 & \sqrt{2} \end{bmatrix}, U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Now  $LY = b$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{4}{\sqrt{2}} & 1 & 0 \\ \sqrt{2} & 3 & \sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\sqrt{2}y_1 = 11 \Rightarrow y_1 = \frac{11}{\sqrt{2}}$$

$$\frac{4}{\sqrt{2}}y_1 + y_2 = 28 \Rightarrow y_2 = 28 - 22 = 6$$

$$\sqrt{2}y_1 + 3y_2 + \sqrt{2}y_3 = 31$$

$$\Rightarrow \sqrt{2}(\frac{11}{\sqrt{2}}) + 3(6) + \sqrt{2}y_3 = 31$$

$$\Rightarrow \sqrt{2}y_3 = 31 - 18 - 11$$

$$= 2$$

$$\Rightarrow y_3 = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$Y = \begin{bmatrix} \frac{11}{\sqrt{2}} \\ 6 \\ \sqrt{2} \end{bmatrix}$$

$$UX = Y$$

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{\sqrt{2}} \\ 6 \\ \sqrt{2} \end{bmatrix}$$

$$\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{3}{\sqrt{2}}x_3 = \frac{11}{\sqrt{2}} \rightarrow (i)$$

$$x_2 + 4x_3 = 6 \rightarrow (ii)$$

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$$\sqrt{2}x_3 = \sqrt{2}$$

$$\Rightarrow x_3 = 1$$

$$\text{From (ii)} \quad x_2 = 6 - 4(1) = 2.$$

$$\text{From (i)} \quad \sqrt{2}x_1 = \frac{11}{\sqrt{2}} - \frac{1}{\sqrt{2}}(2) - \frac{3}{\sqrt{2}}(1)$$

$$\Rightarrow x_1 = \frac{6}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow x_1 = \frac{6}{2} = 3$$

So solution vector is  $\underline{x} = [3, 2, 1]^T$

(Answer).

Question:-

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

(i) Find  $A^{-1}$  by Cholesky's method.

(ii) Hence solve system of equations

$$x_1 + 2x_2 + 6x_3 = 13$$

$$2x_1 + 5x_2 + 15x_3 = 15$$

$$6x_1 + 15x_2 + 46x_3 = 19.$$

Solution:-

$$\text{As } A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$$A = A^T$$

i.e.  $A$  is symmetric. then

$$A = LL^T$$

Where  $L^T = U$

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$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$$\begin{aligned} l_{11}^2 &= 1 & l_{21}l_{11} &= 2 & l_{31}l_{11} &= 6 \\ \Rightarrow l_{11} &= 1 & \Rightarrow l_{21} &= 2 & \Rightarrow l_{31} &= 6 \end{aligned}$$

$$\begin{aligned} l_{21}^2 + l_{22}^2 &= 5 & l_{21}l_{31} + l_{22}l_{32} &= 15 \\ (2)^2 + l_{22}^2 &= 5 & (2)(6) + (1)l_{32} &= 15 \\ l_{22}^2 &= 5-4 & l_{32} &= 15-12 \\ \Rightarrow l_{22} &= 1 & \Rightarrow l_{32} &= 3 \end{aligned}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 46$$

$$(6)^2 + (3)^2 + l_{33}^2 = 46$$

$$l_{33}^2 = 46 - 36 - 9$$

$$l_{33}^2 = 1$$

$$\Rightarrow l_{33} = 1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

As we know that  $11^{-1} \cdot 1^{-1} = I$

Next we calculate  $L^{-1}$ .

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$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ 2l_{11}' + l_{21}' & 2l_{12}' + l_{22}' & 2l_{13}' + l_{23}' \\ 6l_{11}' + 3l_{21}' + l_{31}' & 6l_{12}' + 3l_{22}' + l_{32}' & 6l_{13}' + 3l_{23}' + l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing each element corresponding to matrix we get,

$$\Rightarrow l_{11}' = 1, \quad l_{12}' = 0, \quad l_{13}' = 0$$

$$2l_{11}' + l_{21}' = 0, \quad 2l_{12}' + l_{22}' = 1$$

$$\Rightarrow l_{21}' = -2, \quad \Rightarrow l_{22}' = 1$$

$$2l_{13}' + l_{23}' = 0, \quad 6l_{11}' + 3l_{21}' + l_{31}' = 0$$

$$l_{23}' = 0, \quad 6(1) + 3(-2) + l_{31}' = 0$$

$$\Rightarrow l_{31}' = 0$$

$$6l_{12}' + 3l_{22}' + l_{32}' = 0$$

$$6(0) + 3(1) + l_{32}' = 0$$

$$\Rightarrow l_{32}' = -3$$

$$6l_{13}' + 3l_{23}' + l_{33}' = 1$$

$$6(0) + 3(0) + l_{33}' = 1$$

$$\Rightarrow l_{33}' = 1$$

$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

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$$A = LL^T$$

$$A^{-1} = (LL^T)^{-1}$$

$$= (L^T)^{-1} L^{-1}$$

$$= (L^{-1})^T L^{-1}$$

$$\Rightarrow A^{-1} = (L^{-1})^T L^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -2 & 0 \\ -2 & 1+9 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(Answer).

$$(ii) \quad x_1 + 2x_2 + 6x_3 = 13$$

$$2x_1 + 5x_2 + 15x_3 = 15$$

$$6x_1 + 15x_2 + 46x_3 = 19$$

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} 65 - 30 + 0 \\ -26 + 150 - 57 \\ 0 - 45 + 19 \end{bmatrix} = \begin{bmatrix} 35 \\ 67 \\ -26 \end{bmatrix}$$

$$\mathbf{x} = [35, 67, -26]^T$$

(Answer).

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$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ 2l_{11}' + l_{21}' & 2l_{12}' + l_{22}' & 2l_{13}' + l_{23}' \\ 6l_{11}' + 3l_{21}' + l_{31}' & 6l_{12}' + 3l_{22}' + l_{32}' & 6l_{13}' + 3l_{23}' + l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing each element corresponding to matrix we get,

$$\Rightarrow l_{11}' = 1, \quad l_{12}' = 0, \quad l_{13}' = 0$$

$$2l_{11}' + l_{21}' = 0, \quad 2l_{12}' + l_{22}' = 1$$

$$\Rightarrow l_{21}' = -2, \quad \Rightarrow l_{22}' = 1$$

$$2l_{13}' + l_{23}' = 0, \quad 6l_{11}' + 3l_{21}' + l_{31}' = 0$$

$$l_{23}' = 0, \quad 6(1) + 3(-2) + l_{31}' = 0$$

$$\Rightarrow l_{31}' = 0$$

$$6l_{12}' + 3l_{22}' + l_{32}' = 0$$

$$6(0) + 3(1) + l_{32}' = 0$$

$$\Rightarrow l_{32}' = -3$$

$$6l_{13}' + 3l_{23}' + l_{33}' = 1$$

$$6(0) + 3(0) + l_{33}' = 1$$

$$\Rightarrow l_{33}' = 1$$

$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

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$$A = LL^T$$

$$A^{-1} = (LL^T)^{-1}$$

$$= (L^T)^{-1} L^{-1}$$

$$= (L^{-1})^T L^{-1}$$

$$\Rightarrow A^{-1} = (L^{-1})^T L^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -2 & 0 \\ -2 & 1+9 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(Answer).

$$(ii) \quad x_1 + 2x_2 + 6x_3 = 13$$

$$2x_1 + 5x_2 + 15x_3 = 15$$

$$6x_1 + 15x_2 + 46x_3 = 19$$

$$A \underline{x} = \underline{b}$$

$$\underline{x} = A^{-1} \underline{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} 65 - 30 + 0 \\ -26 + 150 - 57 \\ 0 - 45 + 17 \end{bmatrix} = \begin{bmatrix} 35 \\ 67 \\ -26 \end{bmatrix}$$

$$\underline{x} = [35, 67, -26]^T$$

(Answer).

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# EXERCISE

1): Factorization method apply on matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

in a form  $A=LU$ , where  $L$  is lower triangular and  $U$  is upper triangular matrices.

2): Solve the following system of equation by LU- Decomposition by crout's method.

$$3x_1 + 2x_2 + x_3 = 10$$

$$x_1 + 4x_2 - x_3 = 6$$

$$x_1 + 2x_2 + 5x_3 = 20$$

3): Find inverse of the following matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$

by expressing  $A$  in a form of LU- Decomposition by Doolittle's Method.

4): Solve the system of equations by factorizing the coefficient matrix approximately

$$x_1 + 2x_2 + (-1)x_3 = 5$$

$$2x_1 + 3x_2 - 5x_3 = 13$$

$$3x_1 + 8x_2 + 6x_3 = 13$$

5): Write the matrix  $A = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$

as a product of an upper and a lower triangular matrices and hence invert it.

6): Solve by Cholesky's method of system of equations

$$3x_1 + 2x_2 + x_3 = 10$$

$$x_1 + 4x_2 + x_3 = 6$$

$$2x_1 + x_2 + 4x_3 = 12$$

7): Solve the following system of linear equation

$$2x_1 + 0x_2 + x_3 = 4$$

$$-3x_1 + 4x_2 - 2x_3 = -3$$

$$x_1 + 7x_2 - 5x_3 = 6$$

by

(i) Doolittle's Method

(ii) Crout's Method

(iii) Cholesky's Method.

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# ★ LDU FACTORIZATION :-

LDU-Factorization is an other method to factorize any (square) matrix. To express  $A = LDU$ . Where  $L$  and  $U$  are lower and upper triangular matrices with unit entries in diagonal and  $D$  is a diagonal matrix.

To find inverse by expressing  $A = LDU$  is:

$$A = LDU$$

$$A^{-1} = (LDU)^{-1}$$

$$= U^{-1} D^{-1} L^{-1}$$

$$UA^{-1} = D^{-1} L^{-1}$$

where  $L^{-1}$  is given by  $LL^{-1} = I$ .

then find  $L^{-1}$  we have  $A^{-1}$ .

Example:

Obtain LDU factorization of matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:-

$$LDU = A$$

$$\text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and } D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} d_{11} & d_{11}u_{12} & d_{11}u_{13} \\ d_{11}l_{21} & d_{11}l_{21}u_{12} + d_{22} & d_{11}l_{21}u_{13} + d_{22}u_{23} \\ d_{11}l_{31} & d_{11}l_{31}u_{12} + d_{22}l_{32} & d_{11}l_{31}u_{13} + d_{22}l_{32}u_{23} + d_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

By comparing we have:

$$\Rightarrow d_{11} = 1$$

$$\Rightarrow d_{11}u_{12} = -1$$

$$\Rightarrow u_{12} = -1$$

$$d_{11}l_{21} = 1$$

$$\Rightarrow l_{21} = 1$$

$$d_{11}u_{13} = -2$$

$$\Rightarrow u_{13} = -2$$

$$d_{11}l_{31} = 1$$

$$\Rightarrow l_{31} = 1$$

$$d_{11}l_{21}u_{12} + d_{22} = 2$$

$$1(1)(-1) + d_{22} = 2$$

$$\Rightarrow d_{22} = 3$$

$$d_{11}l_{21}u_{13} + d_{22}u_{23} = 1$$

$$1(1)(-2) + (3)u_{23} = 1$$

$$\Rightarrow u_{23} = 1$$

$$d_{11}l_{31}u_{12} + d_{22}l_{32} = 3$$

$$1(1)(-1) + 3l_{32} = 3$$

$$\Rightarrow l_{32} = 4/3$$

$$d_{11}l_{31}u_{13} + d_{22}l_{32}u_{23} + d_{33} = -1$$

$$1(1)(-2) + 3\left(\frac{4}{3}\right)(1) + d_{33} = -1$$

$$\Rightarrow d_{33} = -3$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$L \quad D \quad U = A$

Example:

Find inverse of matrix  $A$  (define in previous example) by expressing in form of  $LDU$ .

Solution:

(\*) Working Rule:

(i) Express  $A = LDU$

(ii) From  $LL^{-1} = I$  find  $L^{-1}$

(iii) Find  $D^{-1}$

(iv) From  $UA^{-1} = D^{-1}L^{-1}$

Find  $A^{-1} = ?$

In previous example we express  $A = LDU$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Next find  $L^{-1}$

$$LL^{-1} = L^{-1}L = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{11}' + l_{21}' & l_{12}' + l_{22}' & l_{13}' + l_{23}' \\ l_{11}' + \frac{4}{3}l_{21}' + l_{31}' & l_{12}' + \frac{4}{3}l_{22}' + l_{32}' & l_{13}' + \frac{4}{3}l_{23}' + l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} l_{11}' &= 1, & l_{12}' &= 0, & l_{13}' &= 0 \\ l_{11}' + l_{21}' &= 0, & l_{12}' + l_{22}' &= 1, & l_{13}' + l_{23}' &= 0 \\ \Rightarrow l_{21}' &= -1, & \Rightarrow l_{22}' &= 1, & l_{23}' &= 0 \end{aligned}$$

$$\begin{aligned} l_{11}' + \frac{4}{3}l_{21}' + l_{31}' &= 0, & l_{12}' + \frac{4}{3}l_{22}' + l_{32}' &= 0 \\ (1) + \frac{4}{3}(-1) + l_{31}' &= 0, & \frac{4}{3}(1) + l_{32}' &= 0 \\ l_{31}' &= +\frac{1}{3}, & \Rightarrow l_{32}' &= -\frac{4}{3} \end{aligned}$$

$$\begin{aligned} l_{13}' + \frac{4}{3}l_{23}' + l_{33}' &= 1 \\ 0 + \frac{4}{3}(0) + l_{33}' &= 1 \\ \Rightarrow l_{33}' &= 1 \end{aligned}$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/3 & -4/3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$U A^{-1} = D^{-1} L^{-1}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/3 & -1/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}' - a_{21}' - 2a_{31}' & a_{12}' - a_{22}' - 2a_{32}' & a_{13}' - a_{23}' - 2a_{33}' \\ a_{21}' + a_{31}' & a_{22}' + a_{32}' & a_{23}' + a_{33}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & 0 \\ -1/9 & 4/3 & -1/3 \end{bmatrix}$$

$$\Rightarrow a_{31}' = -1/9, \quad a_{23}' + a_{33}' = 0$$

$$\Rightarrow a_{32}' = 4/9, \quad a_{23}' = 1/3$$

$$\Rightarrow a_{33}' = -1/3$$

$$a_{22}' + a_{32}' = 1/3$$

$$\Rightarrow a_{22}' = \frac{1}{3} - \frac{4}{9} = -1/9$$

$$\Rightarrow a_{21}' = -2/9, \quad a_{11}' = 5/9, \quad a_{12}' = 7/9$$

$$\text{and } a_{13}' = -1/3$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 5/9 & 7/9 & -1/3 \\ -2/9 & -1/9 & 1/3 \\ -1/9 & 4/9 & -1/3 \end{bmatrix}$$

(Answer)

INDIRECT METHOD

(OR)

Iterative Method

There are two methods to solve system of linear equations indirectly

- Jacobi Iterative Method
- Gauss-Seidal Method.

(i) Jacobi-Iterative Method:-

Consider the

linear system  $Ax = b$ .

Rearrange the equations so that diagonal elements are not zero and equation are diagonally dominant.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \rightarrow (i)$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \rightarrow (ii)$$

$$a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \rightarrow (iii)$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \rightarrow (n)$$

Find  $x_1$  from eq (i),  $x_2$  from eq (ii),  $x_3$  from eq (iii) and continuing from eq (n) find value of  $x_n$ .

$$x_1 = \frac{1}{a_{11}} (b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n)$$



$$x_n = \frac{1}{a_{nn}} (b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1})$$

In general we write it as

$$x_i = \frac{1}{a_{ii}} \left[ b_i - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j \right], i=1, 2, \dots, n$$

Setting iteration

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)}] \\ x_2^{(k+1)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)}] \\ x_3^{(k+1)} &= \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - \dots - a_{3n}x_n^{(k)}] \\ &\vdots \\ x_n^{(k+1)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \dots - a_{nn-1}x_{n-1}^{(k)}] \end{aligned}$$

In general we write it as setting iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j^{(k)} \right], i=1, 2, 3, \dots, n$$

is the required formula for Jacobi's method. Starting from an initial guess, successive solution vector are computed until convergence has been obtained.

\* Condition of Convergence for a Linear system of equation (Convergence Criteria for Jacobi's method)

The condition of convergence for system of linear equations is that linear system is diagonally dominant. The linear system of equations is said to be diagonally dominant if in each equation, the absolute value of diagonal coefficient is greater than or equal to the sum of absolute values of off-diagonal coefficients and is atleast one equation is actually greater than the sum i.e.

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \text{ for all } i$$

$$\text{and } |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

for atleast one value of  $i$

The condition is sufficient but not necessary condition for convergence. That is, if the coefficient satisfy this condition then method must converge. If not, the convergence is not guaranteed. It may not and probably will not converge.



\* Example: Solve system of equations by Jacobi's Method

$$10x_1 + 3x_2 + x_3 = 19 \rightarrow (i)$$

$$3x_1 + 10x_2 + 2x_3 = 29 \rightarrow (ii)$$

$$x_1 + 2x_2 + 10x_3 = 35 \rightarrow (iii)$$

Solution:- As equations are diagonally dominant

From (i)

$$x_1 = \frac{1}{10} (19 - 3x_2 - x_3)$$

From (ii)

$$x_2 = \frac{1}{10} (29 - 3x_1 - 2x_3)$$

From (iii)

$$x_3 = \frac{1}{10} (35 - x_1 - 2x_2)$$

Setting iteration

$$x_1^{(k+1)} = \frac{1}{10} (19 - 3x_2^{(k)} - x_3^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{10} (29 - 3x_1^{(k)} - 2x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{10} (35 - x_1^{(k)} - 2x_2^{(k)})$$

Let initial guess is

$$x = (0, 0, 0)^T$$

$$\text{i.e. } x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

1<sup>st</sup> Iteration:-

When  $k=0$

$$x_1^{(1)} = \frac{1}{10} (19 - 3x_2^{(0)} - x_3^{(0)})$$

$$= \frac{1}{10} (19 - 3(0) - 0)$$

$$= 1.9$$

$$x_2^{(1)} = \frac{1}{10} (29 - 3x_1^{(0)} - 2x_3^{(0)})$$

$$= \frac{1}{10} (29 - 3(0) - 2(0))$$

$$= \frac{1}{10} 29$$

$$x_3^{(1)} = \frac{1}{10} (35 - x_1^{(0)} - 2x_2^{(0)})$$

$$= \frac{1}{10} (35 - 0 - 2(0))$$

$$= \frac{1}{10} 35$$

$$= 3.5$$

2<sup>nd</sup> Iteration:- when  $k=1$

$$x_1^{(2)} = \frac{1}{10} (19 - 3x_2^{(1)} - x_3^{(1)})$$

$$= \frac{1}{10} (19 - 3(2.9) - (3.5)) = 0.68$$

$$x_2^{(2)} = \frac{1}{10} (29 - 3x_1^{(1)} - 2x_3^{(1)})$$

$$= \frac{1}{10} (29 - 3(1.9) - 2(3.5)) = 1.63$$

$$x_3^{(2)} = \frac{1}{10} (35 - x_1^{(1)} - 2x_2^{(1)})$$

$$= \frac{1}{10} (35 - (1.9) - 2(2.9)) = 2.73$$

3<sup>rd</sup> Iteration

$k=2$

$$x_1^{(3)} = \frac{1}{10} (19 - 3x_2^{(2)} - x_3^{(2)})$$

$$= \frac{1}{10} (19 - 3(1.63) - 2.73) = 1.1380$$

$$x_2^{(3)} = \frac{1}{10} (29 - 3x_1^{(2)} - 2x_3^{(2)})$$

$$= \frac{1}{10} (29 - 3(0.68) - 2(2.73)) = 2.150$$

$$x_3^{(3)} = \frac{1}{10} (35 - x_1^{(2)} - 2x_2^{(2)})$$

$$= \frac{1}{10} (35 - 0.68 - 2(1.63))$$

$$= 3.106$$

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4<sup>th</sup> Iteration:

$$x_1^{(4)} = \frac{1}{10} (19 - 3x_2^{(3)} - x_3^{(3)})$$

$$= \frac{1}{10} (19 - 3(2.150) - 2.106)$$

$$= 0.9444$$

$$x_2^{(4)} = \frac{1}{10} (29 - 3x_1^{(4)} - 2x_3^{(3)})$$

$$= \frac{1}{10} (29 - 3(1.138) - 2(3.106))$$

$$= 1.9374$$

$$x_3^{(4)} = \frac{1}{10} (35 - x_1^{(4)} - 2x_2^{(4)})$$

$$= \frac{1}{10} (35 - 1.138 - 2(2.150))$$

$$= 2.9562$$

5<sup>th</sup> Iteration:

$$x_1^{(5)} = \frac{1}{10} (19 - 3x_2^{(4)} - x_3^{(4)})$$

$$= \frac{1}{10} (19 - 3(1.9374) - 2.9562)$$

$$= 1.0232$$

$$x_2^{(5)} = \frac{1}{10} (29 - 3x_1^{(5)} - 2x_3^{(4)})$$

$$= \frac{1}{10} (29 - 3(0.9444) - 2(2.9562))$$

$$= 2.0254$$

$$x_3^{(5)} = \frac{1}{10} (35 - x_1^{(5)} - 2x_2^{(5)})$$

$$= \frac{1}{10} (35 - 0.9444 - 2(1.9374))$$

$$= 3.0181$$

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6<sup>th</sup> Iteration:When  $k=5$ 

$$x_1^{(6)} = \frac{1}{10} (19 - 3(2.0254) - 3.0181)$$

$$= 0.9906$$

$$x_2^{(6)} = \frac{1}{10} (29 - 3(1.0232) - 2(3.0181))$$

$$= 1.9894$$

$$x_3^{(6)} = \frac{1}{10} (35 - 1.0232 - 2(2.0254))$$

$$= 2.9926$$

7<sup>th</sup> Iteration:When  $k=6$ 

$$x_1^{(7)} = \frac{1}{10} (19 - 3(1.9894) - 2.9926)$$

$$= 1.0039$$

$$x_2^{(7)} = \frac{1}{10} (29 - 3(0.9906) - 2(2.9926))$$

$$= 2.0043$$

$$x_3^{(7)} = \frac{1}{10} (35 - 0.9906 - 2(1.9894))$$

$$= 3.0031$$

8<sup>th</sup> Iteration:When  $k=7$ 

$$x_1^{(8)} = \frac{1}{10} (19 - 3(2.0043) - 3.0031)$$

$$= 0.9984$$

$$x_2^{(8)} = \frac{1}{10} (29 - 3(1.0039) - 2(3.0031))$$

$$= 1.9982$$

$$x_3^{(8)} = \frac{1}{10} (35 - 1.0039 - 2(2.0043))$$

$$= 2.9988$$

9<sup>th</sup> Iteration:When  $k=8$ 

$$x_1^{(9)} = \frac{1}{10} (19 - 3(1.9982) - 2.9988)$$

$$= 1.0007$$

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$$x_1^{(9)} = \frac{1}{10} (29 - 3(0.9984) - 2(2.9988))$$

$$= 2.007$$

$$x_2^{(9)} = \frac{1}{10} (35 - 0.9984 - 2(1.9982))$$

$$= 3.0005$$

10<sup>th</sup> Iteration:When  $k = 9$ 

$$x_1^{(10)} = \frac{1}{10} (19 - 3(2.007) - 3.0005)$$

$$= 0.9979$$

$$x_2^{(10)} = \frac{1}{10} (29 - 3(1.007) - 2(3.0005))$$

$$= 1.9978$$

$$x_3^{(10)} = \frac{1}{10} (35 - 1.007 - 2(2.007))$$

$$= 2.9998$$

11<sup>th</sup> Iteration:When  $k = 10$ .

$$x_1^{(11)} = \frac{1}{10} (19 - 3(1.9978) - 2.9998)$$

$$= 1.000$$

$$x_2^{(11)} = \frac{1}{10} (29 - 3(0.9979) - 2(2.9998))$$

$$= 2.000$$

$$x_3^{(11)} = \frac{1}{10} (35 - 0.9979 - 2(1.9978))$$

$$= 3.000$$

Hence solution vector is  $x = [1, 2, 3]^T$ 

(Answer).

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\* Example:

Consider

$$4x - y + z = 7$$

$$4x - 8y + z = -21$$

$$-2x + y + 5z = 15$$

Solved by Jacobi's Method.

Solution: Re-write these equations we have

$$x = \frac{1}{4} (7 + y - z)$$

$$y = \frac{1}{8} (21 + 4x + z)$$

$$z = \frac{1}{5} (15 + 2x - y)$$

Setting iteration is

$$x^{(k+1)} = \frac{1}{4} (7 + y^{(k)} - z^{(k)})$$

$$y^{(k+1)} = \frac{1}{8} (21 + 4x^{(k)} + z^{(k)})$$

$$z^{(k+1)} = \frac{1}{5} (15 + 2x^{(k)} - y^{(k)})$$

Let Guess vector is  $(x^{(0)}, y^{(0)}, z^{(0)}) = (1, 2, 2)^T$ 

| K | $x^{(k)}$ | $y^{(k)}$ | $z^{(k)}$ |
|---|-----------|-----------|-----------|
| 0 | 1         | 2         | 2         |
| 1 | 1.844     | 3.875     | 3.025     |
| 2 | 1.963     | 3.925     | 2.963     |
| 3 | 1.991     | 3.977     | 3.000     |
| 4 | 1.994     | 3.996     | 3.001     |
| 5 | 1.999     | 3.997     | 2.998     |
| 6 | 2.000     | 3.999     | 3.000     |
| 7 | 2.000     | 4.000     | 3.000     |

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Hence Solution vector is

$$(x, y, z) = (2, 4, 3)^T \quad (\text{Answer})$$

### \* GAUSS-SEIDEL METHOD:

Consider the linear system  $Ax = b$ . Re-arrange equations so that diagonal elements are not zero and equation are diagonally dominant.

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &= b_3 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \right\} (i)$$

Find  $x_1$  from 1<sup>st</sup> equation,  $x_2$  from 2<sup>nd</sup> equation and so on find  $x_n$ . Then system (i) can be written as

$$x_1 = \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n]$$

$$x_2 = \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n]$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}} [b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1}]$$

Setting iteration we get

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)}] \\ x_2^{(k+1)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)}] \\ x_3^{(k+1)} &= \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - \dots - a_{3n}x_n^{(k)}] \\ \vdots \\ x_n^{(k+1)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{nn-1}x_{n-1}^{(k+1)}] \end{aligned}$$

In general it also can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(k)} \right]$$

$i = 1, 2, 3, \dots, n$

Starting from an initial guess, successive solution vectors are computed until convergence have been obtained.

Remark:-

Gauss-Seidel method converges twice as fast as Jacobi's Method.

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\* Example:-

Solve  $10x_1 + 3x_2 + x_3 = 19$

$3x_1 + 10x_2 + 2x_3 = 29$

$x_1 + 2x_2 + 10x_3 = 35$

by Gauss-Seidel Method.

Solution:-

Equations are diagonally dominant.

$$x_1 = \frac{1}{10} [19 - 3x_2 - x_3]$$

$$x_2 = \frac{1}{10} [29 - 3x_1 - 2x_3]$$

$$x_3 = \frac{1}{10} [35 - x_1 - 2x_2]$$

Setting Iteration are

$$x_1^{(k+1)} = \frac{1}{10} [19 - 3x_2^{(k)} - x_3^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{10} [29 - 3x_1^{(k+1)} - 2x_3^{(k)}]$$

$$x_3^{(k+1)} = \frac{1}{10} [35 - x_1^{(k+1)} - 2x_2^{(k+1)}]$$

Initial estimate  $x^0 = (0, 0, 0)^T$ 1<sup>st</sup> Iteration: When  $k=0$ 

$$x_1^{(1)} = \frac{1}{10} [19 - 3(0) - 0] = 1.9$$

$$x_2^{(1)} = \frac{1}{10} [29 - 3(1.9) - 2(0)] = 2.33$$

$$x_3^{(1)} = \frac{1}{10} [35 - 1.9 - 2(0)] = 2.844$$

2<sup>nd</sup> Iteration:When  $k=1$ 

$$x_1^{(2)} = \frac{1}{10} [19 - 3(2.33) - 2.844] = 0.9166$$

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$$x_2^{(2)} = \frac{1}{10} [29 - 3(0.9166) - 2(2.83)] = 2.0562$$

$$x_3^{(2)} = \frac{1}{10} [35 - 0.9166 - 2(2.0562)] = 2.9971$$

3<sup>rd</sup> Iteration:When  $k=2$ 

$$x_1^{(3)} = \frac{1}{10} [19 - 3(2.0562) - 2.9971] = 0.9834$$

$$x_2^{(3)} = \frac{1}{10} [29 - 3(0.9834) - 2(2.9971)] = 2.0056$$

$$x_3^{(3)} = \frac{1}{10} [35 - 0.9834 - 2(2.0056)] = 3.0005$$

4<sup>th</sup> Iteration:When  $k=3$ 

$$x_1^{(4)} = \frac{1}{10} [19 - 3(2.0056) - 3.0005] = 0.9983$$

$$x_2^{(4)} = \frac{1}{10} [29 - 3(0.9983) - 2(3.0005)] = 2.004$$

$$x_3^{(4)} = \frac{1}{10} [35 - 0.9983 - 2(2.0004)] = 3.0001$$

5<sup>th</sup> Iteration:When  $k=4$ 

$$x_1^{(5)} = \frac{1}{10} [19 - 3(2.004) - 3.0001] = 0.999$$

$$x_2^{(5)} = \frac{1}{10} [29 - 3(0.9999) - 2(3.0001)] = 2.000$$

$$x_3^{(5)} = \frac{1}{10} [35 - 0.999 - 2(2.000)] = 3.000$$

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Hence solution vector is

$$x = [1, 2, 3]^T \quad (\text{Answer})$$

\* Example 2 Solve by Gauss-Seidel method

$$10x_1 + x_2 + 2x_3 = 44$$

$$x_1 + 2x_2 + 10x_3 = 61$$

$$2x_1 + 10x_2 + x_3 = 51$$

Solution:-

As equations are not diagonally dominant, rearranging we have

$$10x_1 + x_2 + 2x_3 = 44$$

$$2x_1 + 10x_2 + x_3 = 51$$

$$x_1 + 2x_2 + 10x_3 = 61$$

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$$\begin{aligned} x_1 &= \frac{1}{10} [44 - x_2 - 2x_3] \\ x_2 &= \frac{1}{10} [51 - 2x_1 - x_3] \\ x_3 &= \frac{1}{10} [61 - x_1 - 2x_2] \end{aligned}$$

Iterative equations are

$$x_1^{(k+1)} = \frac{1}{10} [44 - x_2^{(k)} - 2x_3^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{10} [51 - 2x_1^{(k+1)} - x_3^{(k)}]$$

$$x_3^{(k+1)} = \frac{1}{10} [61 - x_1^{(k+1)} - 2x_2^{(k+1)}]$$

Initial vector  $x^{(0)} = (0, 0, 0)^T$

1<sup>st</sup> Iteration:- When  $k=0$

$$x_1^{(1)} = \frac{1}{10} [44 - 0 - 0] = 4.4$$

$$x_2^{(1)} = \frac{1}{10} [51 - 2(4.4) - 0] = 4.22$$

$$x_3^{(1)} = \frac{1}{10} [61 - 4.4 - 2(4.22)] = 4.816$$

2<sup>nd</sup> Iteration:

When  $k=1$

$$x_1^{(2)} = \frac{1}{10} [44 - 4.22 - 2(4.816)] = 3.0148$$

$$x_2^{(2)} = \frac{1}{10} [51 - 2(3.0148) - 4.816] = 4.0154$$

$$x_3^{(2)} = \frac{1}{10} [61 - 3.0148 - 2(4.0154)] = 4.9954$$

3<sup>rd</sup> Iteration:

When  $k=2$

$$x_1^{(3)} = \frac{1}{10} [44 - 4.0154 - 2(4.9954)] = 2.9994$$

$$x_2^{(3)} = \frac{1}{10} [51 - 2(2.9994) - 4.9954] = 4.0007$$

$$x_3^{(3)} = \frac{1}{10} [61 - 2.9994 - 2(4.0007)] = 4.9999$$

4<sup>th</sup> Iteration:

When  $k=3$

$$x_1^{(4)} = \frac{1}{10} [44 - 4.0007 - 2(4.9999)] = 2.9999$$

$$x_2^{(4)} = \frac{1}{10} [51 - 2(2.9999) - 4.9999] = 4.0000$$

$$x_3^{(4)} = \frac{1}{10} [61 - 2.9999 - 2(4.0000)] = 5.0000$$

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5<sup>th</sup> Iteration:When  $k = 4$ 

$$x_1^{(5)} = \frac{1}{10} [44 - 4.000 - 2(5.000)]$$

$$= \frac{3.000}{10}$$

$$x_2^{(5)} = \frac{1}{10} [51 - 2(3.000) - 5.000]$$

$$= \frac{4.000}{10}$$

$$x_3^{(5)} = \frac{1}{10} [61 - 3.000 - 2(4.000)]$$

$$= \frac{5.000}{10}$$

Hence solution vector  $x = (3, 4, 5)^T$ 

(Answer).

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## EXERCISE

(1): Explain Jacobi's Method to solve iterative method.

2): Explain Gauss-Seidel method and write advantages of Gauss-Seidel method.

3): Write general iterative formula for both Jacobi's and Gauss-Seidel methods for solving a system of linear equation.

4): Describe the convergence criterion for a system of linear equation.

5): Solve by Jacobi's Method carry out three iterations. Rearrange equation for convergence

$$5x_1 + 2x_2 + x_3 = 14$$

$$x_1 + x_2 + 8x_3 = 20$$

$$x_1 + 5x_2 - x_3 = 10.$$

6): Apply 3 iteration of Gauss-Seidel method

$$8x_1 + x_2 - x_3 = 8$$

$$3x_1 - x_2 + 9x_3 = 12$$

$$x_1 - 7x_2 + 2x_3 = -4.$$

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7) Apply 4 iteration of Gauss-Seidel method after arranging equations for convergence

$$x_1 + 2x_2 + 5x_3 = 2$$

$$6x_1 - x_2 - x_3 = 3$$

$$2x_1 + 7x_2 - x_3 = -3$$

8) Apply Jacobi's Method

$$4x_1 + x_2 - x_3 = 0$$

$$x_1 - 4x_2 + 2x_3 = 13$$

$$2x_1 + x_2 + 5x_3 = 10$$

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### \* NORM OF A MATRIX :-

If 'A' is an  $n \times n$  matrix then norm of A is real number, denoted by  $\|A\|$  s.t the following conditions are satisfied

$$(i) \|A\| \geq 0$$

$$(ii) \|A\| = 0 \iff A = 0$$

$$(iii) \|K \cdot A\| = |K| \|A\|$$

$$(iv) \|A+B\| \leq \|A\| + \|B\|$$

$$(v) \|AB\| \leq \|A\| \|B\|$$

### Some Useful Norms :-

(i) Column Norm (OR) 1-Norm

$$\|A\|_c = \|A\|_1 = \max_j \sum_i |a_{ij}|$$

Example :

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & 6 \\ +1 & 3 & 9 \end{bmatrix}$$

$$\begin{aligned} \|A\|_1 &= \max(1+3+1, 2+8+3, 5+6+9) \\ &= \max(5, 13, 20) \\ &= 20 \end{aligned}$$

$$\|A\|_1 = 20$$

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(ii) Row Norm (OR)  $\infty$ -Norm.

$$\|A\|_{\infty} = \|A\|_{\infty} = \max_i \sum_j |a_{ij}|$$

Example:  $\|A\|_{\infty} = \text{Max}(1+2+5, 3+8+6, 1+3+9)$   
 $= \text{Max}(8, 17, 13)$   
 $= 17$

(iii) Euclidian Norm (OR) 2-Norm.

$$\|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

Example:

$$\|A\|_2 = \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 + |a_{21}|^2 + |a_{22}|^2 + |a_{23}|^2 + |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2}$$

$$= \sqrt{1 + 4 + 25 + 9 + 64 + 36 + 1 + 9 + 81}$$

$$= \sqrt{230}$$

$$= 15.1658$$

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\* Condition Number:-

If 'A' is a non-singular matrix then condition number of A is denoted and defined by  
 $\text{Cond}(A) = K(A) = \|A\| \|A^{-1}\|$

Example:-

$$\text{If } A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

then find  $K_1(A)$ ,  $K_2(A)$  and  $K_{\infty}(A)$ .

Solution:-

$$\text{As } A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and}$$

$$A^{-1} = \begin{bmatrix} 1 & -1/5 & -2/5 \\ 1 & -1/5 & -3/5 \\ -1 & 2/5 & 4/5 \end{bmatrix}$$

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(i)  $\|A\|_1 = \text{Max}(2+3+1, 0+2+1, 1+5+0)$   
 $= \text{Max}(6, 3, 6)$   
 $= 6$

$$\|A^{-1}\|_1 = \text{Max}(1+1+1, \frac{1}{5} + \frac{1}{5} + \frac{2}{5}, \frac{2}{5} + \frac{7}{5} + \frac{4}{5})$$

$$= \text{Max}(3, 4/5, 13/5)$$

$$= 3$$

$$K_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$$

$$= 6 \times 3 = 18$$



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$$(ii) \|A\|_{\infty} = \text{Max}(2+0+1, 3+2+5, 1+1+0)$$

$$= \text{Max}(3, 10, 2)$$

$$= 10$$

$$\|A^{-1}\|_{\infty} = \text{Max}\left(1+\frac{1}{5}+\frac{2}{5}, 1+\frac{1}{5}+\frac{7}{5}, 1+\frac{2}{5}+\frac{4}{5}\right)$$

$$= \text{Max}(1.600, 2.600, 2.200)$$

$$= 2.6000$$

$$K_{\infty}(A) = \|A\|_{\infty} \cdot \|A^{-1}\|_{\infty}$$

$$= 10 \times 2.6000$$

$$= 26.000$$

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$$(iii) \|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

$$= \sqrt{4+0+1+9+4+25+1+1+0}$$

$$= \sqrt{45}$$

$$\|A^{-1}\|_2 = \sqrt{1+\frac{1}{25}+\frac{4}{25}+\frac{1}{25}+\frac{1}{25}+\frac{49}{25}+\frac{1}{25}+\frac{4}{25}+\frac{16}{25}}$$

$$= \sqrt{3+\frac{75}{25}}$$

$$= \sqrt{6}$$

$$K_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

$$= \sqrt{45} \cdot \sqrt{6}$$

$$= 3\sqrt{30}$$

$$= 16.4317$$

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Theorem:-

Condition number of a matrix is always greater than or equal to one.

Proof:-

Let  $A$  be any square matrix then

$$A \cdot A^{-1} = I$$

$$\text{Now } 1 \leq \|I\|$$

$$= \|A \cdot A^{-1}\|$$

$$\leq \|A\| \cdot \|A^{-1}\|$$

$$= \text{Cond}(A)$$

$$\Rightarrow 1 \leq \text{Cond}(A)$$

$$\text{or } \text{Cond}(A) \geq 1$$

(proved).

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\* Importance of Condition Number:-

Condition

number of matrix is a measure of condition of the matrix. Large condition number of matrix indicates ill condition of the system of linear equation  $AX = b$ . Small condition number of matrix  $A$  indicates the system is well condition or Stable.

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Example:-

Consider the system

$$1.01x_1 + 0.99x_2 = 2$$

$$0.99x_1 + 1.01x_2 = 2$$

then  $x = [1, 1]^T$  is its solution but if it is written as

$$1.01x_1 + 0.99x_2 = 2$$

$$0.99x_1 + 1.01x_2 = -2$$

then  $x = [100, -100]^T$  is its solution. Determine it is stable or ill condition.

Solution:-

$$\text{Here } A = \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}$$

$$|A| = 1.0201 - 0.9801$$

$$|A| = 0.0400$$

$$A^{-1} = \begin{bmatrix} 25.250 & -24.75 \\ -24.75 & 25.25 \end{bmatrix}$$

$$\begin{aligned} \|A\|_1 &= \max(1.01+0.99, 0.99+1.01) \\ &= \max(2, 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \|A^{-1}\|_1 &= \max(25.25+24.75, 24.75+25.25) \\ &= \max(50, 50) \\ &= 50 \end{aligned}$$

$$\begin{aligned} K_1(A) &= \|A\|_1 \cdot \|A^{-1}\|_1 \\ &= 2 \times 50 = 100 \end{aligned}$$

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And

$$\begin{aligned} \|A\|_\infty &= \max(1.01+0.99, 0.99+1.01) \\ &= \max(2, 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \|A^{-1}\|_\infty &= \max(25.25+24.75, 24.75+25.25) \\ &= 50 \end{aligned}$$

$$\begin{aligned} K_\infty(A) &= \|A\|_\infty \cdot \|A^{-1}\|_\infty \\ &= 2 \times 50 \\ &= 100 \end{aligned}$$

Condition number is large so system is ill condition.

Remark:-

A system is supposed to be ill-condition when one or more of the following indicators are presents

- (1) Small change in coefficients of matrix A or in element of b results a large change in solution vector.
- (2) Matrix A possesses a very small determinant as compared to the average of its elements.
- (3) If  $\text{Cond}(A) = \|A\| \|A^{-1}\|$  is large.

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### \* ERROR & RESIDUAL OF AN APPROXIMATION :-

Consider the linear system

$$Ax = b$$

Let  $x$  be the exact solution and  $x^{(a)}$  be approximate solution of system then

$$e = x - x^{(a)}$$

is called error and

$$r = Ax - Ax^{(a)}$$

is called residual of an approximation.

The quantities  $\frac{\|r\|}{\|b\|}$  and  $\frac{\|e\|}{\|x\|}$  are

called relative residual and relative error of the approximation solution  $x^{(a)}$ .

### \* Example:

$$\text{Let } 1.01x_1 + 0.99x_2 = 2 \\ 0.99x_1 + 1.01x_2 = 2$$

then  $x = [1, 1]^T$  is exact solution and

$\tilde{x} = [1.01, 1.01]^T$  is approximate solution then find error and residual.

$$\begin{aligned} \text{Solution: } e &= x - x^{(a)} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.01 \\ 1.01 \end{bmatrix} \\ &= \begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix} \end{aligned}$$

$$\text{And } r = Ax - A\tilde{x}^{(a)}$$

$$\begin{aligned} \Rightarrow r &= b - Ax^{(a)} \\ &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2.02 \\ 2.02 \end{bmatrix} \\ &= \begin{bmatrix} -0.02 \\ -0.02 \end{bmatrix} \end{aligned}$$

Here  $e$  and  $r$  both are small Now if we take

$$x^{(a)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\text{then } e = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and } r = \begin{bmatrix} -0.02 \\ 0.02 \end{bmatrix}$$

Here error is relatively large to the residual but, as the system is ill condition, so all this indicates that error and residual does not indicate whether the system is ill condition or well condition (stable).

### AN IMPORTANT RESULT :-

Now consider  $Ax = b$ , then

$$e = x - x^{(a)}$$

$\therefore x^{(a)}$  is approximate solution

and

$$\begin{aligned} r &= Ax - Ax^{(a)} \\ &= A(x - x^{(a)}) \end{aligned}$$



$$\underline{r} = A \underline{e} \rightarrow (i)$$

$$\Rightarrow \underline{e} = A^{-1} \underline{r} \rightarrow (ii)$$

From (i)

$$\begin{aligned} \|\underline{r}\| &= \|A \underline{e}\| \\ &\leq \|A\| \|\underline{e}\| \rightarrow (iii) \end{aligned}$$

From (ii)

$$\begin{aligned} \|\underline{e}\| &= \|A^{-1} \underline{r}\| \\ &\leq \|A^{-1}\| \|\underline{r}\| \rightarrow (iv) \end{aligned}$$

From (iii)

$$\frac{\|\underline{r}\|}{\|A\|} \leq \|\underline{e}\| \rightarrow (v)$$

From (iv) and (v)

$$\frac{\|\underline{r}\|}{\|A\|} \leq \|\underline{e}\| \leq \|A^{-1}\| \|\underline{r}\| \rightarrow (vi)$$

Now if we choose  $\underline{x}^{(a)} = 0$   
then

$$\underline{r} = A \underline{x} - A \underline{x}^{(a)}$$

$$= A \underline{x} - 0$$

$$= A \underline{x}$$

$$\underline{r} = \underline{b}$$

and

$$\underline{e} = \underline{x} - \underline{x}^{(a)}$$

$$\underline{e} = \underline{x}$$

Then (vi) becomes

$$\frac{\|\underline{b}\|}{\|A\|} \leq \|\underline{x}\| \leq \|A^{-1}\| \|\underline{b}\|$$

$$\Rightarrow \frac{\|A\|}{\|\underline{b}\|} \geq \frac{1}{\|\underline{x}\|} \geq \frac{1}{\|A^{-1}\| \|\underline{b}\|}$$

$$\Rightarrow \frac{1}{\|A^{-1}\| \|\underline{b}\|} \leq \frac{1}{\|\underline{x}\|} \leq \frac{\|A\|}{\|\underline{b}\|} \rightarrow (vii)$$

Multiplying (vi) by (vii)

$$\frac{1}{\|A\| \|A^{-1}\|} \frac{\|\underline{r}\|}{\|\underline{b}\|} \leq \frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \|A\| \|A^{-1}\| \frac{\|\underline{r}\|}{\|\underline{b}\|}$$

$$\Rightarrow \frac{1}{K(A)} \frac{\|\underline{r}\|}{\|\underline{b}\|} \leq \frac{\|\underline{e}\|}{\|\underline{x}\|} \leq K(A) \frac{\|\underline{r}\|}{\|\underline{b}\|} \rightarrow (viii)$$

Now if  $K(A) \approx 1$  i.e. system is stable, then from (viii)

$$\frac{\|\underline{r}\|}{\|\underline{b}\|} \leq \frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \frac{\|\underline{r}\|}{\|\underline{b}\|}$$

$$\Rightarrow \frac{\|\underline{r}\|}{\|\underline{b}\|} \approx \frac{\|\underline{e}\|}{\|\underline{x}\|}$$

Hence if the system is stable then relative residual and relative error are approximately equal.

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## \* Eigen Values & Eigen Vectors:-

If 'A' is a square matrix over  $\mathbb{R}$ , then  $\lambda \in \mathbb{R}$  is said to be eigen value of A if for some non-zero column vector

$$AV = \lambda V$$

Example:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow 5 = \lambda$  is eigen value of that matrix.

Remark:-

If  $AV = \lambda V$ ,  $V \neq 0$ , then  $\lambda$  is called eigen value of A and V is called eigen vector of A corresponding to  $\lambda$ . An eigen value is also called characteristic root or latent root and eigen vector is also called characteristic vector or latent vector.

\* Theorem:-

If  $A$  is invertible and  $AB=0$   
then  $B=0$ .

Proof:- As  $A$  is invertible, so  $A^{-1}$  exists.

Thus  $AB=0$

$$\Rightarrow A^{-1}(AB) = A^{-1} \cdot 0$$

$$\Rightarrow (A^{-1}A)B = 0$$

$$\Rightarrow IB = 0$$

$$\Rightarrow B = 0.$$

(proved)

\* Remark:-

(i) From above theorem if  $B \neq 0$  then  $A$   
is non-invertible.

(ii) Now

$$A V = \lambda V$$

$$\Rightarrow AV - \lambda V = 0$$

$$\Rightarrow (A - \lambda I)V = 0$$

$$\Rightarrow A - \lambda I \text{ is singular } \because V \neq 0$$

$$\Rightarrow |A - \lambda I| = 0.$$

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\* POWER METHOD:-

It is an iterative  
method for finding dominant eigen value  
and eigen vector of a matrix.

(i) Dominant Eigen Value:-

An eigen value  $\lambda$  of  
square matrix  $A$  is said to be dominant  
eigen value if  $|\lambda|$  has largest value as  
compared to absolute value of the other  
eigen value of  $A$ . Then corresponding eigen  
vector is called dominant eigen vector.

(ii) Normalized Eigen Vector:-

An eigen vector  
 $x$  is said to be normalized if the  
co-ordinate of largest magnitude is 1.  
For example

$[1, 0, 0]^T$ ,  $[1, 1, 1]^T$ ,  
 $[0, 1, 1]^T$  are all normalized  
vectors.

WORKING RULE FOR POWER METHOD:-

To find  
dominant eigen value and corresponding  
eigen vector of matrix take  
 $x_0$  is initial vector which is normalized  
vector like  $x_0 = [1, 1, 1]^T$



$Ax_0 = C_1 x_1$  where  $x_1$  is normalized

$Ax_1 = C_2 x_2$  where  $x_2$  is normalized

$Ax_2 = C_3 x_3$  where  $x_3$  is normalized

$Ax_n = C_{n+1} x_{n+1}$  where  $x_{n+1}$  is normalized

then  $\lambda = C_{n+1}$  is dominant eigen value  
and  $x_{n+1}$  is the corresponding eigen vector.

The iterative process will be stopped  
when  $x_{n+1} = x_n$ .

Working Rule for finding absolute least eigen value:-

Step I: If  $\lambda_1$  is the dominant eigen value and  $x_1$  be corresponding dominant eigen vector, then find  $B = A - \lambda_1 I$ .

Step II: Find dominant eigen value of  $B$   
say it is  $\lambda_2$ . (By taking negative common).

Step III: Absolute least eigen value of  $A = \lambda_1 + \lambda_2$ .

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Near Moon CHG, Rwp  
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Example:-

Find dominant and absolute least eigen pair of  $A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

Solution:-

Taking initial guess vector

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T$$

$$Ax_0 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \lambda_1 x_1$$

$$Ax_1 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 9.4 \end{bmatrix} = 9.4 \begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \lambda_2 x_2$$

$$Ax_2 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.925 \\ 9.17 \end{bmatrix} = 9.17 \begin{bmatrix} 0.537 \\ 1 \end{bmatrix} = \lambda_3 x_3$$

$$Ax_3 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.537 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.695 \\ 9.074 \end{bmatrix} = 9.074 \begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \lambda_4 x_4$$

$$Ax_4 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.581 \\ 9.032 \end{bmatrix} = 9.032 \begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \lambda_5 x_5$$

$$Ax_5 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.535 \\ 9.014 \end{bmatrix} = 9.014 \begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \lambda_6 x_6$$

$$Ax_6 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.515 \\ 9.006 \end{bmatrix} = 9.006 \begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \lambda_7 x_7$$

$$Ax_7 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.506 \\ 9.0026 \end{bmatrix} = 9.0026 \begin{bmatrix} 0.500 \\ 1 \end{bmatrix} = \lambda_8 x_8$$

$$A \underline{x}_0 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = 9 \underline{x}_0$$

we see that  $\underline{x}_0 = \underline{x}_1$  so stop iteration  
and  $\lambda = 9$  is dominant eigen value and  
 $[0.5, 1]^T$  is dominant eigen vector.

For absolute least eigen value calculate  
 $B = A - \lambda I$

$\because \lambda$  is dominant eigen value.

$$B = A - 9I$$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

Guess vector  $\underline{x}_0 = [1 \ 1]^T$

$$B \underline{x}_0 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_1 \underline{x}_1$$

$$B \underline{x}_1 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_2 \underline{x}_2$$

$$B \underline{x}_2 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_3 \underline{x}_3$$

So  $\lambda = -5$  is dominant eigen value of  $B$ .

Absolute least eigen value of  $A = \lambda_1 + \lambda_2$   
 $= 9 + (-5)$   
 $= 4$

(Answer)

\* Example:- Using power method to find the  
dominant and smallest eigen value and  
Corresponding eigen vector of matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{by take} \quad \underline{y}_0 = [1, 1, 1]^T$$

Solution:

$$A \underline{y}_0 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix} = C_1 \underline{y}_1$$

$$A \underline{y}_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 1.03 \\ 0.2 \end{bmatrix} = 4.6 \begin{bmatrix} 1 \\ 0.22 \\ 0.04 \end{bmatrix} = C_2 \underline{y}_2$$

$$A \underline{y}_2 = \begin{bmatrix} 4 & 1 \\ 0.22 & 1 \\ 0.04 & -0.04 \end{bmatrix} \begin{bmatrix} 1 \\ 4.22 \end{bmatrix} = \begin{bmatrix} 4.22 \\ 0.11 \\ -0.01 \end{bmatrix} = C_3 \underline{y}_3$$

$$A \underline{y}_3 = \begin{bmatrix} 4 & 1 \\ 0.11 & 1 \\ -0.01 & 0.01 \end{bmatrix} \begin{bmatrix} 1 \\ 4.11 \end{bmatrix} = \begin{bmatrix} 4.11 \\ 0.05 \\ 0.00 \end{bmatrix} = C_4 \underline{y}_4$$

$$A \underline{y}_4 = \begin{bmatrix} 4 & 1 \\ 0.05 & 1 \\ 0.00 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4.05 \end{bmatrix} = \begin{bmatrix} 4.05 \\ 0.03 \\ 0 \end{bmatrix} = C_5 \underline{y}_5$$

$$A \underline{y}_5 = \begin{bmatrix} 4 & 1 \\ 0.03 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4.03 \end{bmatrix} = \begin{bmatrix} 4.03 \\ 0.02 \\ 0 \end{bmatrix} = C_6 \underline{y}_6$$

$$A y_1 = A \begin{bmatrix} 1 \\ 0.02 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.02 \\ 0.04 \\ 0 \end{bmatrix} = 4.02 \begin{bmatrix} 1 \\ 0.01 \\ 0 \end{bmatrix} = C_1 y_1$$

$$A y_2 = A \begin{bmatrix} 1 \\ 0.01 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.01 \\ 0.02 \\ 0 \end{bmatrix} = 4.01 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_2 y_2$$

$$A y_3 = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_3 = C_4$$

$\lambda_1 = 4$  is a dominant eigen value and corresponding eigen vector  $[1, 0, 0]^T$ .  
For a smallest eigen value calculate

$$B = A - \lambda I$$

$\because \lambda$  is dominant eigen value.

$$B = A - 4I = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\text{Let } x_0 = (1, 1, 1)^T$$

$$B x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix} = C_1 x_1$$

$$B x_1 = B \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.6 \\ 5 \end{bmatrix} = -5 \begin{bmatrix} -0.04 \\ -0.12 \\ 1 \end{bmatrix} = C_2 x_2$$

$$B x_2 = B \begin{bmatrix} -0.04 \\ -0.12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.12 \\ 1.24 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.02 \\ -0.25 \\ 1 \end{bmatrix} = C_3 x_3$$

$$B x_3 = B \begin{bmatrix} 0.02 \\ -0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.5 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.05 \\ -0.3 \\ 1 \end{bmatrix} = C_4 x_4$$

$$B x_4 = B \begin{bmatrix} 0.05 \\ -0.3 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 1.06 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.06 \\ -0.32 \\ 1 \end{bmatrix} = C_5 x_5$$

$\lambda_2 = -5$  is dominant eigen value of  $B$ .

Smallest eigen value of  $A = \lambda_1 + \lambda_2$   
 $= 4 + (-5)$   
 $= -1$

corresponding eigen vector is  $(0.06, -0.32, 1)^T$

(Answer)

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\* Example:-

Use power Method to compute the largest eigenvalue and corresponding eigen vector of matrix

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

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Solution:- Let  $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

Take  $y_0 = [1 \ 0 \ 0]^T$  as initial vector  
Then by Power method

$$Ay_0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.50 \end{bmatrix}$$

$$Ay_1 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.80 \\ -0.80 \end{bmatrix}$$

$$Ay_2 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.80 \\ -0.80 \end{bmatrix} = \begin{bmatrix} 5.60 \\ 5.20 \\ -5.20 \end{bmatrix} = 5.60 \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix}$$

$$Ay_3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86 \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix}$$

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$$Ay_4 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96 \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix}$$

$$Ay_5 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$Ay_6 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So dominant or largest eigen value is  $\lambda = 6$   
and corresponding eigen vector is  
 $[1, 1, -1]^T$

(Answer).

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## LINEAR OPERATORS

An operation  $P$  is said to be linear if for all  $f_1, f_2$  and  $\alpha_1, \alpha_2$

$$P(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 P(f_1) + \alpha_2 P(f_2)$$

where  $P$  is defined on space  $f$  of functions and  $f_1, f_2 \in f$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ .

In numerical analysis the following linear operators are defined

- (i) Shift Operator
- (ii) Forward difference Operator
- (iii) Backward difference Operator
- (iv) Central difference Operator
- (v) Average or Mean value Operator.

(i) Shift Operator:-

$$E f(x) = f(x+h)$$

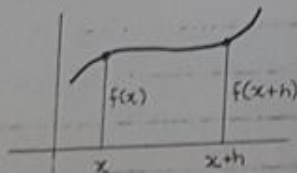
and

$$E^2 f(x) = E(f(x+h)) = f(x+h+h)$$

$$E^2 f(x) = f(x+2h)$$

Similarly; In general

$$E^n f(x) = f(x+nh)$$



(ii) Forward Difference Operator:-

Let the values

$x_0, x_0+h, x_0+2h, \dots, x_0+nh$  be corresponding independent variables and  $f(x_0), f(x_0+h), f(x_0+2h), \dots, f(x_0+nh)$  be corresponding dependent variable of function  $f(x)$ .

The values of  $x$  are known as argument and  $f(x)$  is called entries.

where  $h$  be a common difference between intervals.

Now Forward difference operator is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

which is of order one.

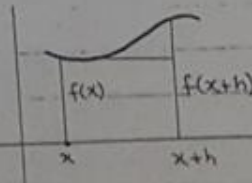
Similarly;  $\Delta f(x+h) = f(x+2h) - f(x+h)$

$$\Delta f(x+nh) = f(x+(n+1)h) - f(x+nh)$$

In 2<sup>nd</sup> order forward difference defined as

$$\begin{aligned} \Delta^2 f(x) &= \Delta [f(x+h) - f(x)] \\ &= f(x+2h) - f(x+h) - [f(x+h) - f(x)] \\ &= f(x+2h) - f(x+h) - f(x+h) + f(x) \\ &= f(x+2h) - 2f(x+h) + f(x) \end{aligned}$$

Similarly;  $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$



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(iii) Backward Difference operator:-

It is

denoted and defined as

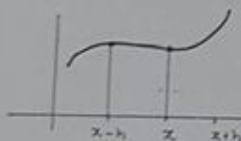
$$\nabla f(x) = f(x) - f(x-h)$$

$$\text{and } \nabla f(x+h) = f(x+h) - f(x)$$

$$\nabla f(x+2h) = f(x+2h) - f(x+h)$$

$$\nabla f(x+rh) = f(x+rh) - f(x+(r-1)h)$$

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(iv) Central Difference Operator:-

The first

order central difference operator is denoted and defined as

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$

(v) Average Operator:-Average operator denoted by  $\mu$  and defined by

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

2

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\* RELATIONS WITH SHIFT OPERATOR:-(i) Relation between Forward difference operator and Shift Operator:-

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$= (E - 1) f(x)$$

$$\Rightarrow \Delta f(x) = (E - 1) f(x)$$

$$\Rightarrow \Delta = E - 1$$

$$\Rightarrow E = \Delta + 1$$

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(ii) Relation between Backward difference operator and Shift Operator:-

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$= (1 - E^{-1}) f(x)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = (1 - E^{-1})$$

$$\Rightarrow E^{-1} = 1 - \nabla$$

$$\Rightarrow E = (1 - \nabla)^{-1}$$

(iii) Relation between Central difference operator and Shift operator:-

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta^2 = (E^{1/2} - E^{-1/2})^2$$



$$\delta^2 = E + E^{-1} - 2$$

(i) Relation between Average Operator and Shift Operator:

$$M f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

$$= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$$

$$M f(x) = \frac{(E^{1/2} + E^{-1/2}) f(x)}{2}$$

$$M = \frac{E^{1/2} + E^{-1/2}}{2}$$

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Formulae :

$$\Delta = E - 1$$

$$E = 1 + \Delta$$

$$\nabla = 1 - E^{-1}$$

$$E = (1 - \nabla)^{-1}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$M = \frac{1}{2} [E^{1/2} + E^{-1/2}]$$

\* Question:-

Prove the following Identities

(i)  $E = e^{hD}$  ;  $D = d/dx$ .

$$E f(x) = f(x+h)$$

$$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \dots$$

$$E f(x) = \left( 1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots \right) f(x)$$

$$\Rightarrow E = 1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots$$

$$E = e^{hD} \quad \text{where } D = \frac{d}{dx} \quad (\text{proved}).$$

(ii)  $\Delta \sin k = 2 \sin(h/2) \cos(k+h/2)$ .

L.H.S  $\Delta \sin k$

$$= \sin(k+h) - \sin k$$

$$= 2 \cos\left(\frac{k+h+k}{2}\right) \sin\left(\frac{k+h-k}{2}\right)$$

$$= 2 \cos\left(\frac{2k+h}{2}\right) \sin(h/2)$$

$$= 2 \sin(h/2) \cos(k+h/2) \quad (\text{proved}).$$

(iii)  $\Delta - \nabla = \Delta \nabla = \nabla \Delta = \delta^2$

$$\Delta \nabla = (E-1)(1-E^{-1})$$

$$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1} \rightarrow (i)$$

$$\Delta - \nabla = (E-1) - (1-E^{-1})$$

$$= E - 1 - 1 + E^{-1}$$

$$= E - 2 + E^{-1} \rightarrow (ii)$$

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$$\Rightarrow \delta^2 = E + E^{-1} - 2$$

(iv) Relation between Average Operator and Shift Operator:

$$\begin{aligned} \mu f(x) &= \frac{f(x+h/2) + f(x-h/2)}{2} \\ &= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2} \end{aligned}$$

$$\mu f(x) = \frac{(E^{1/2} + E^{-1/2}) f(x)}{2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

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Formulae :

$$\Delta = E - 1$$

$$E = 1 + \Delta$$

$$\nabla = 1 - E^{-1}$$

$$E = (1 - \nabla)^{-1}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{1}{2} [E^{1/2} + E^{-1/2}]$$

\* Question:-

Prove the following Identities

(i)  $E = e^{hD}$ ;  $D = d/dx$ .

$$E f(x) = f(x+h)$$

$$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \dots$$

$$E f(x) = \left( 1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots \right) f(x)$$

$$\Rightarrow E = 1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots$$

$$E = e^{hD} \quad \text{where } D = \frac{d}{dx} \quad (\text{Proved}).$$

(ii)  $\Delta \sin k = 2 \sin(h/2) \cos(k+h/2)$ .

$$\text{L.H.S. } \Delta \sin k$$

$$= \sin(k+h) - \sin k$$

$$= 2 \cos\left(\frac{k+h+k}{2}\right) \sin\left(\frac{k+h-k}{2}\right)$$

$$= 2 \cos\left(\frac{2k+h}{2}\right) \sin(h/2)$$

$$= 2 \sin(h/2) \cos(k+h/2) \quad (\text{Proved}).$$

(iii)  $\Delta - \nabla = \Delta \nabla = \nabla \Delta = \delta^2$

$$\Delta \nabla = (E-1)(1-E^{-1})$$

$$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1} \rightarrow (i)$$

$$\Delta - \nabla = (E-1) - (1-E^{-1})$$

$$= E - 1 - 1 + E^{-1}$$

$$= E - 2 + E^{-1} \rightarrow (ii)$$

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$$\nabla \Delta = (1 - E^{-1})(E - 1)$$

$$= E - 1 - 1 + E^{-1}$$

$$= E - 2 + E^{-1} \rightarrow (iii)$$

$$\delta^2 = (E^{1/2} - E^{-1/2})^2$$

$$= E + E^{-1} - 2 \rightarrow (iv)$$

From (i), (ii), (iii), and (iv) (proved).

$$(iv) \sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$$

$$\Delta^2 f_k = \Delta(\Delta f_k)$$

$$= \Delta(f_{k+1} - f_k)$$

$$\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k \rightarrow (i)$$

Put  $k = 0, 1, 2, \dots, n-1$  in eq (i)

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$$

$$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$$

$$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$$

$$\vdots$$

$$\vdots$$

$$\Delta^2 f_{n-1} = \Delta f_n - \Delta f_{n-1}$$

Adding

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0 \quad (\text{proved}).$$

$$(v) \Delta^n e^{ax+b} = e^{ax+b} (e^{ah} - 1)^n$$

$$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$

$$= e^{ax+b} (e^{ah} - 1) \rightarrow (i)$$

$$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$$

$$= \Delta(e^{ax+b} (e^{ah} - 1))$$

$\therefore$  From (i)

$$= (e^{ah} - 1)(\Delta e^{ax+b})$$

$$= (e^{ah} - 1) \{ e^{a(x+h)+b} (e^{ah} - 1) \}$$

$$= e^{ax+b} (e^{ah} - 1)^2$$

Continuing in this way we have

$$\Delta^n e^{ax+b} = e^{ax+b} (e^{ah} - 1)^n \quad (\text{proved}).$$

$$(vi) \frac{\Delta^2}{E} (\sin x) = 2 \sin x (\cosh - 1).$$

E Taking L.H.S

$$\frac{\Delta^2}{E} (\sin x) = \Delta^2 E^{-1} (\sin x)$$

$$= \Delta^2 \sin(x-h)$$

$$= \Delta(\Delta \sin(x-h))$$

$$= \Delta \{ \sin x - \sin(x-h) \}$$

$$= \Delta \sin x - \Delta \sin(x-h)$$

$$= \sin(x+h) - \sin x - \sin x + \sin(x-h)$$

$$= \sin(x+h) - 2 \sin x + \sin(x-h)$$

$$= \frac{2 \sin(x+h+x-h) \cos(x+h-x-h)}{2} - 2 \sin x$$

$$= 2 \sin x \cosh - 2 \sin x$$

$$= 2 \sin x (\cosh - 1)$$

$$= R.H.S. \quad (\text{proved}).$$

$$(vii) \Delta(f_k g_k) = f_{k+1} \Delta g_k + g_k \Delta f_k.$$

$$\text{L.H.S } \Delta(f_k g_k)$$

$$= f_{k+1} g_{k+1} - f_k g_k$$

$$= f_{k+1} g_{k+1} - f_{k+1} g_k + f_{k+1} g_k - f_k g_k$$

$$= f_{k+1} (g_{k+1} - g_k) + (f_{k+1} - f_k) g_k$$



$$= f_{k+1} \Delta g_k + g_k \Delta f_k$$

$$= R.H.S.$$

(proved).

$$(vii) \Delta \left( \frac{f_k}{g_k} \right) = \frac{g_k \Delta f_k - f_k \Delta g_k}{g_k g_{k+1}}$$

$$L.H.S. \Delta \left( \frac{f_k}{g_k} \right) = \frac{f_{k+1}}{g_{k+1}} - \frac{f_k}{g_k}$$

$$= \frac{f_{k+1} g_k - f_k g_{k+1}}{g_k g_{k+1}}$$

$$= \frac{f_{k+1} g_k - f_k g_{k+1}}{g_k g_{k+1}}$$

$$= \frac{f_{k+1} g_k - f_k g_{k+1}}{g_k g_{k+1}}$$

$$= \frac{g_{k+1} \cdot g_k}{g_k g_{k+1}} = R.H.S.$$

$$= \frac{g_k \Delta f_k - f_k \Delta g_k}{g_{k+1} \cdot g_k} \quad (\text{proved}).$$

$$(ix) hD = \log(1+\Delta) = -\log(1-\nabla)$$

$$\text{As we know that } e^{hD} = E$$

$$\Rightarrow hD = \log(E)$$

$$hD = \log(1+\Delta) \rightarrow (i)$$

$$\text{Again } e^{hD} = E$$

$$\Rightarrow hD = \log(E)$$

$$hD = \log(1-\nabla)^{-1}$$

$$hD = -\log(1-\nabla) \rightarrow (ii)$$

From (i) and (ii)

$$hD = \log(1+\Delta) = -\log(1-\nabla)$$

(proved).

$$(x) \Delta^k f_i = \sum_{r=0}^k (-1)^{k+r} \binom{k}{r} f_{i+r}$$

$$L.H.S. = \Delta^k f_i = (E-1)^k f_i$$

$$= (-1)^k (1-E)^k f_i$$

$$= (-1)^k \left\{ \sum_{r=0}^k \binom{k}{r} (1)^{k-r} (-E)^r \right\} f_i$$

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \quad \text{by binomial Theorem.}$$

$$= \left\{ \sum_{r=0}^k (-1)^k \binom{k}{r} (-1)^r E^r \right\} f_i$$

$$= \sum_{r=0}^k (-1)^{k+r} \binom{k}{r} f_{i+r}$$

$$\therefore E^n f(x) = f(x+nh)$$

$$= R.H.S.$$

$$\Rightarrow E^r f_i = f_{i+r}$$

$$(xi) \Delta^2 y_k = \nabla^2 y_{k+2}$$

$$L.H.S. = \Delta^2 y_k$$

$$= (E-1)^2 y_k$$

$$= \left( \sum_{i=0}^2 \binom{2}{i} E^{2-i} (-1)^i \right) y_k$$

$$= \left( \sum_{i=0}^2 (-1)^i \binom{2}{i} E^{2-i} \right) y_k$$

$$= \sum_{i=0}^2 (-1)^i \binom{2}{i} y_{k+2-i}$$

$$R.H.S. = \nabla^2 y_{k+2}$$

$$= (1-E^{-1})^2 y_{k+2}$$

$$= \left( \sum_{i=0}^2 \binom{2}{i} (1)^{2-i} (-E^{-1})^i \right) y_{k+2}$$

$$= \left( \sum_{i=0}^2 (-1)^i \binom{2}{i} E^{-i} \right) y_{k+2}$$

$$= \sum_{i=0}^2 (-1)^i \binom{2}{i} y_{k+2-i}$$

$$L.H.S. = R.H.S.$$

(proved).

$$(xi) \quad e^x = \frac{\Delta^2 e^x \cdot E e^x}{E \Delta^2 e^x}$$

$$R.H.S = \frac{\Delta^2 e^x \cdot E e^x}{E \Delta^2 e^x}$$

$$= \frac{\Delta^2 E^{-1} e^x \cdot E e^x}{\Delta^2 e^x}$$

$$= \frac{\Delta^2 e^{x-h} \cdot E e^x}{\Delta^2 e^x}$$

$$= \frac{e^{-h} \Delta^2 e^x \cdot E e^x}{\Delta^2 e^x}$$

$$= e^{-h} E e^x$$

$$= e^{-h} e^{x+h} \quad (\because E f(x) = f(x+h))$$

$$= e^x = L.H.S \quad (\text{proved})$$

$$(xii) \quad \mu\delta = \frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{1}{2} (\Delta + \nabla)$$

$$\mu\delta = (E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \frac{E - E^{-1}}{2} \quad \text{--- (i)} \quad \because (x-a)(x+a) = x^2 - a^2$$

$$\frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{\Delta E^{-1} + \Delta}{2}$$

$$= \frac{-(1-E)E^{-1} + (E-1)}{2}$$

$$= \frac{1 - E^{-1} + E - 1}{2} = \frac{E - E^{-1}}{2} \rightarrow \text{(ii)}$$

$$\frac{1}{2} (\Delta + \nabla) = \frac{1}{2} (E - E^{-1} + E^{-1} - E^{-1})$$

$$= \frac{E - E^{-1}}{2} \rightarrow \text{(iii)}$$

From (i), (ii) and (iii)

$$\mu\delta = \frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{1}{2} (\Delta + \nabla) \quad (\text{proved})$$

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$$(xiv) \quad \Delta - \frac{1}{2} \delta^2 = \delta \sqrt{1 + \delta^2/4}$$

$$L.H.S = \Delta - \frac{1}{2} \delta^2$$

$$= E - 1 - \frac{1}{2} (E^{1/2} - E^{-1/2})^2$$

$$= E - 1 - \frac{1}{2} (E + E^{-1} - 2)$$

$$= E - 1 - \frac{E}{2} - \frac{E^{-1}}{2} + 1$$

$$= \frac{E - E^{-1}}{2}$$

$$R.H.S = \delta \sqrt{1 + \frac{\delta^2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{E + E^{-1} + 2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{(2)^2}}$$

$$= (E^{1/2} - E^{-1/2}) (E^{1/2} + E^{-1/2})$$

$$= \frac{E - E^{-1}}{2}$$

$$L.H.S = R.H.S \quad (\text{proved})$$

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$$(vi) \quad e^x = \frac{\Delta^2 e^x}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$R.H.S = \frac{\Delta^2 e^x}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$= \frac{\Delta^2 E^{-1} e^x}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$= \frac{\Delta^2 e^{x-h}}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$= e^{-h} \Delta^2 e^x \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$= e^{-h} E e^x$$

$$= e^{-h} e^{x+h} \quad (\because E f(x) = f(x+h))$$

$$= e^x = L.H.S \quad (\text{proved})$$

$$(viii) \quad H\delta = \frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{1}{2} (\Delta + \nabla)$$

$$H\delta = (E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \frac{E - E^{-1}}{2} \quad \text{--- (i)} \quad \because (x-a)(x+a) = x^2 - a^2$$

$$\frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{\Delta E^{-1} + \Delta}{2}$$

$$= \frac{(1-E)E^{-1} + (E-1)}{2}$$

$$= \frac{1-E^{-1}+E-1}{2} = \frac{E-E^{-1}}{2} \rightarrow \text{(ii)}$$

$$\frac{1}{2} (\Delta + \nabla) = \frac{1}{2} (E - I + I - E^{-1})$$

$$= \frac{E - E^{-1}}{2} \rightarrow \text{(iii)}$$

From (i), (ii) and (iii)

$$H\delta = \frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{1}{2} (\Delta + \nabla) \quad (\text{proved})$$

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$$(xiv) \quad \Delta - \frac{1}{2} \delta^2 = \delta \sqrt{1 + \delta^2/4}$$

$$L.H.S = \Delta - \frac{1}{2} \delta^2$$

$$= E - 1 - \frac{1}{2} (E^{1/2} - E^{-1/2})^2$$

$$= E - 1 - \frac{1}{2} (E + E^{-1} - 2)$$

$$= E - 1 - \frac{E}{2} + \frac{E^{-1}}{2} + 1$$

$$= \frac{E - E^{-1}}{2}$$

$$R.H.S = \delta \sqrt{1 + \frac{\delta^2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{E + E^{-1} + 2}{4}}$$

$$= (E^{1/2} - E^{-1/2}) \sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{(2)^2}}$$

$$= (E^{1/2} - E^{-1/2}) \frac{(E^{1/2} + E^{-1/2})}{2}$$

$$= \frac{E - E^{-1}}{2}$$

$$L.H.S = R.H.S \quad (\text{proved})$$

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$$(xv) \quad hD = \sinh^{-1}(H\delta)$$

Rewrite problem we have

$$\begin{aligned} \sinh(hD) &= H\delta \\ &= \frac{(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})}{2} \\ &= \frac{E - E^{-1}}{2} \quad \therefore E = e^{hD} \\ &= \frac{e^{hD} - e^{-hD}}{2} \\ &= \sinh(hD) \quad \text{proved.} \end{aligned}$$

$$(xvi) \quad H = (1 - \nabla)(1 - \nabla)^{-1/2}$$

R.H.S

$$\begin{aligned} &= \frac{(1 - \nabla)(1 - \nabla)^{-1/2}}{2} \\ &= \frac{2 - (1 - E^{-1})}{2} \cdot \left\{ \frac{1 - (1 - E^{-1})}{2} \right\}^{-1/2} \\ &= \frac{2 - 1 + E^{-1}}{2} (1 - 1 + E^{-1})^{-1/2} \\ &= \frac{1 + E^{-1}}{2} (E^{1/2}) = \frac{E^{1/2} + E^{-1/2}}{2} = H \\ &= \text{R.H.S} \\ &(\text{proved}). \end{aligned}$$

$$(xvii) \quad (E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = 2 + \Delta$$

L.H.S

$$\begin{aligned} &= (E^{1/2} + E^{-1/2})(1 + E - 1)^{1/2} \\ &= (E^{1/2} + E^{-1/2})(E)^{1/2} \\ &= E + E^{-1/2+1/2} \\ &= 1 + \Delta + E^0 \\ &= 1 + \Delta + 1 \\ &= 2 + \Delta \\ &= \text{R.H.S} \quad (\text{proved}). \end{aligned}$$

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$$(xviii) \quad K(f_k g_k) = K f_k K g_k + \frac{1}{4} \delta f_k \delta g_k.$$

$$\begin{aligned} \text{L.H.S } K(f_k g_k) &= \frac{E^{1/2} + E^{-1/2}}{2} (f_k g_k) \\ &= \frac{1}{2} \{ E^{1/2} (f_k g_k) + E^{-1/2} (f_k g_k) \} \\ &= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}. \end{aligned}$$

$$\text{R.H.S} = K f_k K g_k + \frac{1}{4} \delta(f_k g_k)$$

$$\begin{aligned} &= \left( \frac{E^{1/2} + E^{-1/2}}{2} \right) (f_k) \cdot \left( \frac{E^{1/2} + E^{-1/2}}{2} \right) g_k + \frac{1}{2} (E^{1/2} - E^{-1/2}) f_k g_k \\ &= \left( \frac{1}{2} E^{1/2} f_k + \frac{1}{2} E^{-1/2} f_k \right) \left( \frac{1}{2} E^{1/2} g_k + \frac{1}{2} E^{-1/2} g_k \right) + \\ &\quad \left( \frac{1}{2} E^{1/2} f_k - \frac{1}{2} E^{-1/2} f_k \right) \left( \frac{1}{2} E^{1/2} g_k - \frac{1}{2} E^{-1/2} g_k \right) \\ &= \left( \frac{1}{2} f_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} \right) \left( \frac{1}{2} g_{k+\frac{1}{2}} + \frac{1}{2} g_{k-\frac{1}{2}} \right) + \\ &\quad \left( \frac{1}{2} f_{k+\frac{1}{2}} - \frac{1}{2} f_{k-\frac{1}{2}} \right) \left( \frac{1}{2} g_{k+\frac{1}{2}} - \frac{1}{2} g_{k-\frac{1}{2}} \right). \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{4} f_{k+\frac{1}{2}} g_{k-\frac{1}{2}} + \frac{1}{4} f_{k-\frac{1}{2}} g_{k+\frac{1}{2}} + \\ &\quad \frac{1}{4} (f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}) + \frac{1}{4} (f_{k+\frac{1}{2}} g_{k+\frac{1}{2}}) - \frac{1}{4} (f_{k+\frac{1}{2}} g_{k-\frac{1}{2}}) \\ &\quad - \frac{1}{4} (f_{k-\frac{1}{2}} g_{k+\frac{1}{2}}) + \frac{1}{4} (f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}) \end{aligned}$$

$$= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}$$

$$\text{L.H.S} = \text{R.H.S} \quad (\text{proved}).$$

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## EXERCISE

Show that:-

- (i)  $\Delta \log x_k = \log(1 + h/x_k)$
- (ii)  $\nabla \log x_k = -\log(1 - h/x_k)$
- (iii)  $\Delta \sqrt{f_k} = \frac{\Delta f_k}{\sqrt{f_k} + \sqrt{f_{k+1}}}$
- (iv)  $(1 + \Delta)(1 - \nabla) = 1$
- (v)  $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$
- (vi)  $\Delta \left( \frac{1}{f(x)} \right) = -\frac{\Delta f(x)}{f(x)f(x+1)}$
- (vii)  $\frac{\Delta}{E} \sin x = 2 \cos(x - \frac{h}{2}) \sin(\frac{h}{2})$
- (viii)  $\Delta = \delta E^{1/2} = E^{1/2} \delta$
- (ix)  $\mu^2 = 1 + \frac{\delta^2}{4}$
- (x)  $\mu \left( \frac{f_k}{g_k} \right) = \frac{\mu(f_k)\mu(g_k) - \frac{1}{4}\delta(f_k)\delta(g_k)}{g_{k-\frac{1}{2}} \cdot g_{k+\frac{1}{2}}}$
- (xi)  $\Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2} = \delta$

## INTERPOLATION

### \* Definition:-

It is the technique of estimating the value of a function  $y = f(x)$  for any intermediate value of the independent variable 'x' when the value of function corresponding to a number of values of x are given.

Example:

if given for function  $y = f(x)$

|      |   |   |     |    |    |
|------|---|---|-----|----|----|
| x    | 0 | 1 | 2   | 3  | 4  |
| f(x) | 3 | 7 | 5.6 | 11 | 15 |

Then to find  $f(x)$  ;  $0 \leq x < 4$  is called interpolation.

### \* Extrapolation:-

The process of estimating the value of function  $y = f(x)$  outside the range of given values of variable is called extrapolation.

Example: From above example to calculate  $f(-0.05)$  or  $f(4.5)$  etc is called extrapolation.



### \* Arguments and Entries:

If for a function  $y = f(x)$  for value of independent variable  $x$  say  $x_0, x_1, \dots, x_n$ . It is given  $f(x_0), f(x_1), \dots, f(x_n)$ , then  $x_0, x_1, \dots, x_n$  are called arguments while  $f(x_0), f(x_1), \dots, f(x_n)$  are called entries.

### \* Equally Spaced Arguments:

If  $x_0, x_1, \dots, x_n$  are arguments for the variable  $x$  s.t.  $x_k - x_{k-1} = h \quad \forall k = 1, 2, \dots, n$ , then these arguments are said to be equally spaced arguments. where  $h$  is called step size or interval difference.

### \* Unequally Spaced Arguments:

If the arguments are not equally spaced, then they called unequally spaced.

### \* Interpolating Polynomial :-

Suppose we are given the values of function  $f(x)$  at certain points  $x_0, x_1, x_2, \dots, x_n$  and we want to estimate  $f(\alpha)$  for some  $\alpha \in [x_0, x_n]$ . Then we assume that function can be approximated by a polynomial. This polynomial is called interpolating polynomial.

### \* Forward difference formula :-

we know that

$$\Delta f(x) = f(x+h) - f(x)$$

then we note that

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) \\ &= \Delta(f(x+h) - f(x)) \\ &= f(x+2h) - f(x+h) - f(x+h) + f(x) \\ &= f(x+2h) - 2f(x+h) + f(x). \end{aligned}$$

If we denote  $f(x_0)$  by  $y_0$ ,  $f(x_1)$  by  $y_1$ ,  $\dots$ ,  $f(x_n)$  by  $y_n$  then

$$\Delta y_0 = y_1 - y_0$$

$$\begin{aligned} \Delta^2 y_0 &= \Delta(\Delta y_0) \\ &= \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \end{aligned}$$

$$\begin{aligned} \Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta(\Delta y_1 - \Delta y_0) \\ &= \Delta^2 y_{+1} - \Delta^2 y_0 \end{aligned}$$

in general

$$\Delta y_r = y_{r+1} - y_r$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

$$\Delta^3 y_r = \Delta^2 y_{r+1} - \Delta^2 y_r$$

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r.$$



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Forward difference Table :-

| $x$   | $y$   | $\Delta$                 | $\Delta^2$                               | $\Delta^3$                                   | $\Delta^4$                                   |
|-------|-------|--------------------------|--|--|--|
| $x_0$ | $y_0$ |                          |  |  |  |
|       |       | $\Delta y_0 = y_1 - y_0$ |  |  |  |
| $x_1$ | $y_1$ |                          | $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ |  |  |
|       |       | $\Delta y_1 = y_2 - y_1$ |  | $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ |  |
| $x_2$ | $y_2$ |                          | $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ |  | $\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$ |
|       |       | $\Delta y_2 = y_3 - y_2$ |  | $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$ |  |
| $x_3$ | $y_3$ |                          | $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ |  |  |
|       |       | $\Delta y_3 = y_4 - y_3$ |  |  |  |
| $x_4$ | $y_4$ |                          |  |  |  |

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## Interpolating Polynomial

Types

Equally Spaced Data

(Methods)

- i) Newton's Formulae  
(Forward & Backward)
- ii) Gauss Formulae  
(Forward & Backward)
- iii) Stirling's Formula
- iv) Bessel's Formula
- v) Everett's Formula

Unequally spaced data

(Methods)

- i) Aitken's Formula
- ii) Lagrange's Formula
- iii) Newton's divided difference Formula

Remark:-

i) We use formulae of unequally spaced data to calculate equally spaced data interpolating polynomial but equally spaced formulae only applicable on equally spaced data.

ii) Newton's Difference Formulae is also called Gregory newton's difference formulae.

# EQUALLY SPACED DATA

(i) Newton's Forward Formula:-

if  $x_0, x_1, \dots, x_n$  are equally spaced, then for any  $x$ ,

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

Where  $x_0 = 1^{st}$  argument

$$u = \frac{x - x_0}{h}$$

$h =$  is stepsize

Proof:-

Given  $u = \frac{x - x_0}{h}$

$$\Rightarrow uh = x - x_0$$

$$\Rightarrow x = x_0 + uh$$

$$f(x) = f(x_0 + uh)$$

$$= E^u f(x_0) \quad \because E f(x) = f(x+h)$$

$$= (1 + \Delta)^u f(x_0) \quad \because E = 1 + \Delta$$

$$f(x) = \left[ 1 + u\Delta + \frac{u(u-1)}{2!} \Delta^2 + \frac{u(u-1)(u-2)}{3!} \Delta^3 + \dots \right] f(x_0)$$

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

(proved)

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Note:-

This formula is used when estimated value lies in beginning of given data.

Question:-

$$\sin 45^\circ = 0.707, \sin 50^\circ = 0.7760, \sin 55^\circ = 0.8192$$

$$\sin 60^\circ = 0.8660, \text{ Find } \sin 52^\circ \text{ by}$$

Newton's forward formula.

Solution:-

| $x$ | $y$    | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|-----|--------|------------|--------------|--------------|
| 45  | 0.707  |            |              |              |
| 50  | 0.7760 | 0.0589     |              |              |
| 55  | 0.8192 | 0.0532     | -0.0057      |              |
| 60  | 0.8660 | 0.0468     | -0.0064      | -0.0007      |

$$x_0 = 45, x = 52, h = 5$$

$$u = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

By Newton's Forward difference Formula

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

$$f(52) = 0.707 + (1.4)(0.0589) + \frac{(1.4)(0.4)(-0.0057)}{2!}$$

$$+ \frac{(1.4)(0.4)(-0.0064)(-0.0007)}{3!}$$

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$$f(52) = 0.7880$$

(Answer)

Question:-

$$3\sqrt{29} = 3.0727, 3\sqrt{30} = 3.1074$$

$$3\sqrt{27} = 3.000, 3\sqrt{28} = 3.0367 \text{ Find } 3\sqrt{26}$$

using N.F.D. Formula.

$$\text{Solution:- As } f(x) = 3\sqrt{x}$$

| x  | f(x)   | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|----|--------|---------------|-----------------|-----------------|
| 27 | 3.00   |               |                 |                 |
|    |        | 0.0367        |                 |                 |
| 28 | 3.0367 |               | -0.0011         |                 |
|    |        | 0.0358        |                 | 0               |
| 29 | 3.0727 |               | -0.0011         |                 |
|    |        | 0.0347        |                 |                 |
| 30 | 3.1074 |               |                 |                 |

$$\text{As } x = 26, x_0 = 27, h = 1$$

$$u = \frac{x - x_0}{h} = \frac{26 - 27}{1} = -1$$

By Newton's forward difference formula:

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) +$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

$$f(26) = 3.00 + (-1)(0.0367) + \frac{(-1)(-2)}{2!}(-0.0011)$$

$$+ \frac{(-1)(-2)(-3)}{3!}(0)$$

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$$\Rightarrow 3\sqrt{26} = 3.00 - 0.0367 - 0.0011$$

$$= 2.9620$$

(Ans.)

Question:-

Find polynomial which satisfies following data

| x    | 1 | 2  | 3 | 4  |
|------|---|----|---|----|
| f(x) | 1 | -1 | 1 | -1 |

Solution:

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|------|---------------|-----------------|-----------------|
| 1 | 1    |               |                 |                 |
|   |      | -2            |                 |                 |
| 2 | -1   |               | 4               |                 |
|   |      | 2             |                 | 8               |
| 3 | 1    |               | -4              |                 |
|   |      | -2            |                 |                 |
| 4 | -1   |               |                 |                 |

$$x_0 = 1$$

x is a Variable

$$h = 1$$

$$u = \frac{x - x_0}{h} = \frac{x - 1}{1} = x - 1$$

By Newton's Forward difference Formula:

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) +$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

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$$\begin{aligned}
 f(x) &= \frac{1}{1!} + (x-1)(-2) + \frac{(x-1)(x-2)(4)}{2!} + \\
 &\quad \frac{(x-1)(x-2)(x-3)(-8)}{3!} \\
 &= \frac{1}{3} - 2x + 2 + 2(x^2 - 2x - x + 2) + \\
 &\quad - \frac{4}{3}(x^3 - 6x^2 + 11x - 6) \\
 &= \frac{1}{3} - 2x + 2 + 2x^2 - 4x - 2x + 8 - \\
 &\quad \frac{4}{3}x^3 + 8x^2 - \frac{44}{3}x + 8 \\
 &= -\frac{4}{3}x^3 + 10x^2 - \frac{68}{3}x + 15 \\
 &= \frac{1}{3}(-4x^3 + 30x^2 - 68x + 45)
 \end{aligned}$$

(Answer)

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### \*1) Newton's Backward Difference Formula:

If for a function  $f$ , its value at  $x_0, x_1, \dots, x_n$  are  $f(x_0), f(x_1), \dots, f(x_n)$ , then for any value of  $x$

$$\begin{aligned}
 f(x) &= f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) + \\
 &\quad \frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n) + \dots
 \end{aligned}$$

Proof:- Now  $u = \frac{x - x_n}{h}$

where  $x_n$  is last argument and  $h$  is interval difference.

$$\Rightarrow x = x_n + uh$$

$$f(x) = f(x_n + uh)$$

$$= E^u f(x_n)$$

$$= [(1 - \nabla)^{-1}]^u f(x_n)$$

$$= (1 - \nabla)^{-u} f(x_n) \quad \because E = (1 - \nabla)^{-1}$$

$$\begin{aligned}
 &= \left[ 1 + u \nabla + \frac{u(u+1)}{2!} \nabla^2 + \frac{u(u+1)(u+2)}{3!} \nabla^3 \right. \\
 &\quad \left. + \dots \right] f(x_n)
 \end{aligned}$$

$$\Rightarrow f(x) = f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) +$$

$$+ \frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n) + \dots$$

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Note:-

This formula is used when estimating value lies at the end of the given data.

Question:-

The population of tun is given below

Estimate population for year 1965.

| Year   | 1931 | 1941 | 1951 | 1961 | 1971 |
|--------|------|------|------|------|------|
| Popul. | 46   | 66   | 81   | 93   | 101  |

Solution:-

| X    | Y   | 1 <sup>st</sup> DIFF. | 2 <sup>nd</sup> DIFF. | 3 <sup>rd</sup> DIFF. | 4 <sup>th</sup> DIFF. |
|------|-----|-----------------------|-----------------------|-----------------------|-----------------------|
| 1931 | 46  | 20                    |                       |                       |                       |
| 1941 | 66  |                       | -5                    |                       |                       |
| 1951 | 81  | 15                    |                       | 2                     |                       |
| 1961 | 93  | 12                    | -3                    |                       | -3                    |
| 1971 | 101 | 8                     | -4                    | -1                    |                       |

$$x_n = 1971$$

$$x = 1965$$

$$h = 10$$

$$u = \frac{x - x_n}{h} = \frac{1965 - 1971}{10} = \frac{-6}{10} = -0.6$$

By Newton's Backward diff. formula

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$$f(x) = f(x_n) + u(\nabla f(x_n)) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) +$$

$$\frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n) + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 f(x_n)$$

+ .....

$$f(1965) = 101 + (-0.6)(8) + \frac{(-0.6)(-0.4)}{2!} (-4)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!} (-1) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+1)(-0.6+3)}{4!} (-3)$$

$$= 101 - 4.8 + 0.48 + 0.0560 + 0.1008$$

$$= 96.837$$

(Answer)

Question:-

By Constructing difference table, find 7<sup>th</sup> term as well as general term of Sequence 0, 0, 2, 6, 12, 20.

Solution:-

It is easy to calculate general term by using Newton's forward or Newton's backward difference formula.

The given difference table is

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| x | y  | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|----|------------|--------------|--------------|
| 1 | 0  |            |              |              |
| 2 | 0  | 0          | 2            |              |
| 3 | 2  | 2          | 2            | 0            |
| 4 | 6  | 4          | 2            | 0            |
| 5 | 12 | 6          | 2            | 0            |
| 6 | 20 | 8          |              |              |

By Newton's forward difference table

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) +$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots$$

$$\begin{aligned} f(x) &= 0 + (x-1)(0) + \frac{(x-1)(x-2)}{2!} (2) + 0 \\ &= 0 + 0 + (x-2)(x-1) \\ &= x^2 - 3x + 2 \end{aligned}$$

which is general term of sequence.

To find 7<sup>th</sup> term put  $x=7$

$$f(x) = x^2 - 3x + 2$$

$$f(7) = 49 - 21 + 2 = 30.$$

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By Newton's Backward Diff. formula:-

$$x_n = 6$$

$$h = 1$$

$$u = \frac{x - x_n}{h} = \frac{x - 6}{1}$$

$$f(x) = f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) +$$

$$\frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n) + \dots$$

$$= 20 + (x-6)(8) + \frac{(x-6)(x-5)}{2!} (2)$$

$$+ \frac{(x-6)(x-5)(x-4)}{3!} (0)$$

$$= 20 + 8x - 48 + \frac{x^2 - 11x + 30}{2}$$

$$= x^2 - 3x + 2$$

Which is general term of sequence.

For 7<sup>th</sup> term

$$\text{put } x = 7$$

$$f(7) = (7)^2 - 3(7) + 2$$

$$= 49 - 21 + 2$$

$$= 30$$

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(Answer)

Question:-

Find a polynomial which satisfies  
 $y = |x|$  at  $x = -2, -1, 0, 1, 2$

by Newton's backward diff. formula.



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Solution:-

| x  | y | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|----|---|------------|--------------|--------------|--------------|
| -2 | 2 |            |              |              |              |
| -1 | 1 | -1         |              |              |              |
| 0  | 0 | -1         | 0            |              |              |
| 1  | 1 | 1          | +2           | 2            |              |
| 2  | 2 | 1          | 0            | -2           | 4            |

$$h = 1, x_n = 2$$

$$u = \frac{x - x_n}{h} = \frac{x - 2}{1}$$

By Newton's backward diff. Formula

$$f(x) = f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n) + \dots$$

$$= 2 + (x-2)(1) + \frac{(x-2)(x-1)(0)}{2!} + \dots$$

$$+ \frac{(x-2)(x-1)(x)(-2)}{3!} + \dots$$

$$\frac{(x-2)(x-1)(x)(x+1)(-4)}{4!} + \dots$$

$$= 2 + x - 2 - \frac{1}{3}x^3 + x^2 - \frac{2}{3}x - \frac{1}{6}x^4 + \frac{1}{3}x^3$$

$$+ \frac{1}{6}x^2 - \frac{1}{3}x$$

$$f(x) = \frac{-1}{6}x^4 + \frac{7}{6}x^2 \quad (\text{Required polynomial})$$

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\* Central Difference Table:-

Central difference is defined

as

$$\delta(f(x)) = f(x+h/2) - f(x-h/2)$$

or

$$\delta f(x) = f(x+h) - f(x-h)$$

In the case when stepsize is  $2h$  instead of  $h$ .

Central Difference Table is

| x        | f(x)     | 1 <sup>st</sup> Diff. | 2 <sup>nd</sup> Diff. | 3 <sup>rd</sup> Diff. | 4 <sup>th</sup> Diff. |
|----------|----------|-----------------------|-----------------------|-----------------------|-----------------------|
| $x_{-2}$ | $y_{-2}$ | $\Delta y_{-2}$       |                       |                       |                       |
| $x_{-1}$ | $y_{-1}$ |                       | $\Delta^2 y_{-2}$     |                       |                       |
| $x_0$    | $y_0$    | $\Delta y_{-1}$       |                       | $\Delta^3 y_{-2}$     |                       |
|          |          | $\Delta y_0$          | $\Delta^2 y_{-1}$     |                       | $\Delta^4 y_{-2}$     |
| $x_1$    | $y_1$    |                       | $\Delta^2 y_0$        | $\Delta^3 y_{-1}$     |                       |
| $x_2$    | $y_2$    | $\Delta y_1$          |                       |                       |                       |

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# \* (II) GAUSS CENTRAL DIFFERENCE FORMULAE:-

(i) Gauss forward formula:-

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!} \Delta^4 y_{-2} + \dots$$

where  $u = \frac{x - x_0}{h}$

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Proof:-

By Newton's Forward difference formula:

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \dots \rightarrow (i)$$

$$\begin{aligned} \text{Now as } \Delta^{n+1} y_{i-1} &= \Delta(\Delta^n y_{i-1}) \\ &= \Delta^n (y_{i-1+1} - y_{i-1}) \\ &= \Delta^n (y_i - y_{i-1}) \\ \Delta^{n+1} y_{i-1} &= \Delta^n y_i - \Delta^n y_{i-1} \\ \Delta^n y_i &= \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1} \end{aligned}$$

From above

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

and

$$\begin{aligned} \Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ &= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2} \end{aligned}$$

Using all these values in (i) we have

$$\begin{aligned} f(x) &= y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} [\Delta^2 y_{-1} + \Delta^3 y_{-1}] + \\ &\quad \frac{u(u-1)(u-2)}{3!} [\Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}] + \dots \\ &= y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \left[ \frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!} \right] \Delta^3 y_{-1} \\ &\quad + \dots \\ &= y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} \\ &\quad + \dots \end{aligned}$$

which is required Gauss forward formula for equal interval.

Note:-

Gauss forward formula is used when no. of entries below the line (central line) is greater than above central line.

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\* Question:- Find value of  $y$  corresponding to  $x=5$  from table

$x : 2 : 4 : 6 : 8 : 10$   
 $y : 0 : 0 : 1 : 0 : 0$   
 by Gauss forward formula.

Solution:-

The difference table is given by

|          | $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|----------|-----|-----|------------|--------------|--------------|--------------|
| $x_2$    | 2   | 0   |            |              |              |              |
|          | 4   | 0   | 0          |              |              |              |
| $x_{-1}$ | → 5 |     | 1          | 1            |              |              |
| $x_0$    | 6   | 1   |            | -2           | -3           |              |
|          |     |     | -1         |              | 3            |              |
| $x_1$    | 8   | 0   |            | 1            |              |              |
|          |     |     | 0          |              |              |              |
| $x_2$    | 10  | 0   |            |              |              |              |

central line

$$\text{put } u = \frac{x - x_0}{h} = \frac{x - 6}{2}$$

$$\text{at } x=5, u = \frac{5-6}{2} = -0.5$$

By Gauss Forward Formula:

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!} \Delta^4 y_{-2}$$

$$f(5) = 1 + \frac{(-0.5)(-1)}{2} + \frac{(-0.5)(-1.5)(-2)}{3!} + \frac{(-0.5)(-1.5)(0.5)(3)}{4!} + \frac{(-0.5)(-1.5)(0.5)(-2.5)(6)}{5!}$$

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$$f(5) = 1 + 0.5 - 0.75 + 0.19 - 0.23 = 0.71$$

(Answer).

\* Question:-

Find value of  $y$  corresponding to  $x=4.7$  from table by Gauss Forward Formula.

$x : 3 : 4 : 5 : 6$   
 $y : 6 : 24 : 60 : 120$

Solution:-

The difference table is given by:

|          | $x$   | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|----------|-------|-----|------------|--------------|--------------|
| $x_{-1}$ | 3     | 6   |            |              |              |
|          | 4     | 24  | 18         |              |              |
| $x_0$    | → 4.7 |     |            | 18           |              |
|          |       |     | 36         |              | 6            |
| $x_1$    | 5     | 60  |            | 24           |              |
|          |       |     | 60         |              |              |
| $x_2$    | 6     | 120 |            |              |              |

central line

$$u = \frac{x - x_0}{h} = \frac{x - 4}{1}$$

$$\text{At } x=4.7, u = 4.7 - 4 = 0.7$$

By Gauss Forward Formula:

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} + \dots$$

$$f(4.7) = 24 + (0.7)(36) + \frac{(0.7)(-0.3)(18)}{2!} + \frac{(0.7)(-0.3)(1.7)(6)}{3!}$$

$$= 46.95 \quad (\text{Answer})$$

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(ii) Gauss Backward Formula :-

$$f(x) = y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2!} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!} \Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!} \Delta^4 y_{-2} + \dots$$

where  $u = \frac{x - x_0}{h}$ .

Proof:- By Newton's Forward difference formula

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \dots$$

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$$\begin{aligned} \text{As } \Delta^{n+1} y_{i-1} &= \Delta(\Delta^n y_{i-1}) \\ &= \Delta^n(\Delta y_{i-1}) \\ &= \Delta^n(y_{i+1} - y_{i-1}) \\ &= \Delta^n y_i - \Delta^n y_{i-1} \end{aligned}$$

$$\Delta^n y_i = \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1}$$

$$\Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\begin{aligned} \text{and } \Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ &= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2} \end{aligned}$$

using all these value (also put  $\Delta y_0$ 's value).

$$f(x) = y_0 + u [\Delta y_{-1} + \Delta^2 y_{-1}] + \frac{u(u-1)}{2!} [\Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2}] + \frac{u(u-1)(u-2)}{3!} (\Delta^3 y_{-2} + 2\Delta^4 y_{-2} + \Delta^5 y_{-3} + \Delta^6 y_{-3}) + \dots$$

$$= y_0 + u \Delta y_{-1} + \left[ u + \frac{u(u-1)}{2!} \right] \Delta^2 y_{-1} + \left[ \frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!} \right] \Delta^3 y_{-2} + \dots$$

$$= y_0 + u \Delta y_{-1} + u \left[ 1 + \frac{u-1}{2!} \right] \Delta^2 y_{-1} + \frac{u(u-1)}{2!} \left[ 1 + \frac{u-2}{3} \right] \Delta^3 y_{-2} + \dots$$

$$= y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2!} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!} \Delta^3 y_{-2} + \dots$$

is required formula.

Question:-

Find the population of town for 1974 of data by gauss backward Formula.

Year: 1939 : 1949 : 1959 : 1969 : 1979 : 1989  
Popul: 12 : 15 : 20 : 27 : 39 : 52.

Solution:-

|               | x    | y  | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---------------|------|----|----------|------------|------------|------------|------------|
|               | 1939 | 12 |          |            |            |            |            |
|               |      |    | 3        |            |            |            |            |
|               | 1949 | 15 |          | 2          |            |            |            |
|               |      |    | 5        |            | 0          |            |            |
|               | 1959 | 20 |          | 2          |            | 3          |            |
|               |      |    | 7        |            | 3          |            | -10        |
| $x_0$         | 1969 | 27 |          | 5          |            | -7         |            |
|               |      |    | 12       |            | -4         |            |            |
| $\rightarrow$ | 1974 |    |          | 1          |            |            |            |
|               | 1979 | 39 |          | 13         |            |            |            |
|               |      |    |          |            |            |            |            |
|               | 1989 | 52 |          |            |            |            |            |



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Taking  $x_0 = 1969$ ,  $x = 1974$ ,  $h = 10$   
 $u = \frac{x - x_0}{h} = \frac{1974 - 1969}{10} = 0.5$

By Gauss backward Formula

$$f(x) = y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2!} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!} \Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u+2)}{4!} \Delta^4 y_{-2} + \dots$$

$$f(1969) = 27 + 0.5(7) + \frac{(0.5)(1.5)(5)}{2!} + \frac{(0.5)(1.5)(-0.5)(3)}{3!} + \frac{(0.5)(1.5)(-0.5)(2.5)(-7)}{4!} + \frac{(0.5)(1.5)(-0.5)(2.5)(-1.5)(-10)}{5!}$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$

$$= 32.345$$

(Answer).

Question:

Find polynomial satisfying value

$$x: 2 : 4 : 6 : 8 : 10$$

$$y: -2 : 1 : 3 : 8 : 20$$

by using Gauss Formulae.

Solution:-

The difference table is

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| $x$ | $y$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|-----|-----|----------|------------|------------|------------|------------|
| 2   | -2  | 3        |            |            |            |            |
| 4   | 1   |          | -1         |            |            |            |
| 6   | -3  | 2        | 3          | 4          |            |            |
| 8   | 8   | 5        | 7          | 4          |            |            |
| 10  | 20  | 12       |            |            |            |            |

$$u = \frac{x - x_0}{h} = \frac{x - 6}{2} \rightarrow (i)$$

(i) Gauss Forward Formula:

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{1}{3!} u(u-1)(u+1) \Delta^3 y_0 + \frac{u(u-1)(u+1)(u-2)}{4!} \Delta^4 y_{-2} + \dots$$

$$f(x) = 3 + u(5) + \frac{u(u-1)(3)}{2} + \frac{1}{6} u(u^2-1)(4) + 0$$

$$= \frac{2}{3} u^3 + \frac{3}{2} u^2 + \frac{17}{6} u + 3.$$

using (i)

$$f(x) = \frac{2}{3} \frac{(x-6)^3}{8} + \frac{3}{2} \frac{(x-6)^2}{4} + \frac{17}{6} \frac{(x-6)}{2} + 3$$

$$= \frac{1}{12} (x^3 - 18x^2 + 108x - 216) + \frac{3}{8} (x^2 + 36 - 12x) + \frac{17}{12} (x-6) + 3$$



$$f(x) = \frac{1}{12}x^3 - \frac{9}{8}x^2 + \frac{71}{12}x - 10$$

required (Ans)

(ii) Gauss Backward Formula:-

$$f(x) = y_0 + u\Delta y_{-1} + \frac{1}{2}u(u+1)\Delta^2 y_{-1} + \frac{1}{3!}u(u+1)(u-1)\Delta^3 y_{-1} + \dots$$

Put  $u = \frac{x-6}{2}$

$$f(x) = 3 + u(2) + \frac{1}{2}(u+u)(3) + \frac{1}{6}u(u-1)(4)$$

$$= 3 + 2u + \frac{3}{2}u^2 + \frac{2}{3}u^3 - \frac{2}{3}u$$

$$f(x) = \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3$$

$$= \frac{2}{3} \frac{(x-6)^3}{8} + \frac{3}{2} \frac{(x-6)^2}{4} + \frac{17}{6} \frac{(x-6)}{2} + 3$$

$$f(x) = \frac{1}{12}x^3 - \frac{9}{8}x^2 + \frac{71}{12}x - 10$$

which is required polynomial.

(Answer)

## EXERCISE

(i) Given Data, Find interpolation polynomial using

- Newton's forward and
- Newton's backward Difference formula.

Also find  $f(0.25)$ 

| x | 0.1 | 0.2  | 0.3  | 0.4 | 0.5  |
|---|-----|------|------|-----|------|
| y | 1.4 | 1.56 | 1.76 | 2.0 | 2.28 |

Ans:  $2x^2 + x + 1.2$

$(f(0.25) = 1.655)$

(2) Estimate number of students for 1953 from data given below

| Year            | 1948 | 1950 | 1952 | 1954 |
|-----------------|------|------|------|------|
| No. of Students | 50   | 79   | 102  | 113  |

(3) Find polynomial fitting following data

| x | -3 | -2 | -1 | 0 | 1  |
|---|----|----|----|---|----|
| y | 16 | 7  | 4  | 1 | -8 |

Hence calculate y corresponding  $x = -1.5$

(4) Write  $y = \sin \frac{\pi x}{2}$  as polynomial where

value consider with at  $x = 0, 1, 2, 3$ .

- (5) Use Newton's Forward D. Formula to produce a third degree polynomial satisfy following data.

|      |    |    |    |      |
|------|----|----|----|------|
| x    | 0  | 2  | 4  | 6    |
| f(x) | 40 | 48 | 88 | -224 |

- (6) Interpolate by means of Gauss's Forward formula for  $f(32)$  given

$$f(25) = 0.2707$$

$$f(30) = 0.3027$$

$$f(35) = 0.3386$$

$$f(40) = 0.3794$$

$$\text{Ans: } 0.3166$$

- (7) The pressure of wind 'p' corresponding to Velocity 'v' is given by following data Estimate when  $v = 25$  find  $p = ?$

|   |     |    |     |     |
|---|-----|----|-----|-----|
| V | 10  | 20 | 30  | 40  |
| P | 1.1 | 2  | 4.4 | 7.9 |

$$\text{Ans: } 3.0375$$

### (III) Stirling's Formula :-

$$f(x) = y_0 + u \left[ \frac{\Delta y_0 + \Delta y_{-1}}{2} \right] + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} \left[ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{u^2(u^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

$$\text{where } u = \frac{x - x_0}{h}$$

Proof:-

By Gauss's forward Formula

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} + \dots \rightarrow (1)$$

By Gauss's backward Formula

$$f(x) = y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2!} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!} \Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!} \Delta^4 y_{-2} + \dots \rightarrow (2)$$

Taking Mean of (1) and (2)

$$f(x) = y_0 + u \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} \left[ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{u^2(u^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

which is required formula.



\* Question:-

If  $y_{20} = 512$ ,  $y_{30} = 439$ ,  $y_{40} = 346$   
 $y_{50} = 243$  Find  $y_{35}$  by Stirling's  
 Formula.

Solution:-

|       | x  | y   | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|-------|----|-----|------------|--------------|--------------|
| $x_1$ | 20 | 512 |            |              |              |
|       |    |     | -73        |              |              |
| $x_0$ | 30 | 439 |            | -20          |              |
|       |    |     | -93        |              | 10           |
| $x_1$ | 40 | 346 |            | -10          |              |
|       |    |     | -103       |              |              |
| $x_2$ | 50 | 243 |            |              |              |

$$u = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$$

By Stirling's Formula:

$$f(x) = y_0 + u \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \dots$$

$$= 439 + (0.5) \left( \frac{-73 - 93}{2} \right) + \frac{(0.5)^2}{2!} (-20)$$

$$= 439 - 41.5 - 2.3$$

$$f(x) = 395$$

(Answer.)

\* Question:-

Using Stirling's formula to find  $y(28)$   
 given that

| X | 20    | 25    | 30    | 35    | 40    |
|---|-------|-------|-------|-------|-------|
| Y | 49225 | 48316 | 47236 | 45926 | 44306 |

Solution:-

The difference table is

| X  | Y     | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|----|-------|----------|------------|------------|------------|
| 20 | 49225 |          |            |            |            |
|    |       | -909     |            |            |            |
| 25 | 48316 |          | -171       |            |            |
|    |       | -1080    |            | -65        |            |
| 30 | 47236 |          | -230       |            | 9          |
|    |       | -1310    |            | -74        |            |
| 35 | 45926 |          | -310       |            |            |
|    |       | -1620    |            |            |            |
| 40 | 44306 |          |            |            |            |

By Stirling's formula

$$u = \frac{x - 30}{5} = \frac{28 - 30}{5} = -0.4$$

$$f(x) = y_0 + u \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2 - 1)}{3!} \left( \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right)$$

$$+ \frac{1}{4!} u^2 (u^2 - 1) \Delta^4 y_{-2}$$

$$= 47236 + (-0.4) \left( \frac{-1080 - 1310}{2} \right) +$$

$$\frac{(-0.4)^2}{2!} (-230) + \frac{(-0.4)((-0.4)^2 - 1)}{3!} \left( \frac{-65 - 74}{2} \right)$$

$$+ \frac{1}{4!} (-0.4)^2 (0.160 - 1) (-9)$$



$$= 47236 + 478 - 18.4 - 3892 + 0.504$$

$$f(28) \approx 47692$$

(Answer):

Exercise

(i) Use Stirling formula estimate  $y(2.38)$   
From table.

| x    | 2.0     | 2.2     | 2.4    | 2.6    | 2.8    |
|------|---------|---------|--------|--------|--------|
| f(x) | -1.7843 | -0.5837 | 0.0123 | 1.3976 | 1.9798 |

(ii) Using Stirling formula find polynomial  
Satisfy following data.

| x    | 20   | 30   | 40   | 50   |
|------|------|------|------|------|
| f(x) | 1313 | 1727 | 2392 | 3493 |

(iii) Using Stirling formula find  $\tan 16^\circ$ .

| $\theta$      | 0      | 5°     | 10°    | 15°    | 20°    | 25°    |
|---------------|--------|--------|--------|--------|--------|--------|
| $\tan \theta$ | 0      | 0.0875 | 0.1763 | 0.2679 | 0.3640 | 0.4663 |
|               | 30°    |        |        |        |        |        |
|               | 0.5774 |        |        |        |        |        |

Ans: 0.2867

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(IV) BESSEL'S FORMULA :-

$$f(x) = \frac{1}{2}(y_0 + y_1) + (u - \frac{1}{2})\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_0 + \Delta^2 y_{-1})$$

$$+ \frac{u(u-1)(u-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \dots$$

PROOF:- As we know that Gauss Forward Formula.

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \dots \rightarrow (i)$$

And Gauss Backward Formula is -

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \dots \rightarrow (ii)$$

Shifting origin from  $x_0$  to  $x_1$  in (ii):  
also  $u$  will be changed.

$$\text{As } u = \frac{x - x_0}{h}$$

$$\text{If } u = \frac{x - x_1}{h}$$

$$u = \frac{x - (x_0 + h)}{h} = \frac{x - x_0}{h} - 1$$

$$u = \frac{x - x_0}{h} - 1$$

$$u = u - 1$$

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Value of  $u$  will be changed to  $u-1$ .

Put in (ii) we get

$$f(x) = y_1 + (u-1)\Delta y_0 + \frac{(u-1)u}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_1 + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \dots \rightarrow (iii)$$

Taking mean of (i) and (iii)

$$f(x) = \frac{y_0 + y_1}{2} + (u - \frac{1}{2})\Delta y_0 + \frac{u(u-1)}{2!} \left( \frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2} \right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!} \left[ \frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2} \right] + \dots$$

which is required Bessel's formula.

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Near Post Graduate College  
Asghar Mall Rawalpindi  
Hello: 051-4410622 - 4411983

Question:-

Let  $y_{20} = 2854$ ,  $y_{24} = 3162$ ,  $y_{28} = 3544$ ,  $y_{32} = 3992$ . Solve by Bessel's formula taking origin at 24. Find  $y_{25}$ .

Solution:-

|          | X  | Y    | $\Delta Y$ | $\Delta^2 Y$ | $\Delta^3 Y$ |
|----------|----|------|------------|--------------|--------------|
| $x_{-1}$ | 20 | 2854 |            |              |              |
|          |    |      | 308        |              |              |
| $x_0$    | 24 | 3162 |            | 74           |              |
|          |    |      | 382        |              |              |
| $x_1$    | 28 | 3544 |            | 66           |              |
|          |    |      | 448        |              |              |
| $x_2$    | 32 | 3992 |            |              |              |

$$u = \frac{x - x_0}{h}$$

$$u = \frac{25 - 24}{4} = \frac{1}{4} = 0.25$$

By Bessel's formula

$$f(x) = \frac{y_0 + y_1}{2} + (u - \frac{1}{2})\Delta y_0 + \frac{u(u-1)}{2!} \left( \frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2} \right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \dots$$

$$f(25) = \frac{3162 + 3544}{2} + \left( \frac{1}{4} - \frac{1}{2} \right) (382) + \frac{(0.25)(0.25-1)}{2!} \left( \frac{74 + 66}{2} \right) + \frac{(0.25)(0.25-1)(1/4-1/2)}{3!} (8)$$



$$f(25) = 3353 - 95.5 - 6.5625 - 0.0625$$

$$= 3250.875$$

$$\Rightarrow y_{25} = 3250.875$$

(Answer).

Question:-

Use Bessel's formula to find a polynomial satisfying values

$$x : 2 : 4 : 6 : 8 : 10 : 12$$

$$y : -2 : 1 : 3 : 8 : 20 : 32$$

Solution:-

The difference table is

| x  | y  | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|----|----|------------|--------------|--------------|--------------|--------------|
| 2  | -2 |            |              |              |              |              |
|    |    | 3          |              |              |              |              |
| 4  | 1  |            | -1           |              |              |              |
|    |    | 2          |              | 4            |              |              |
| 6  | 3  |            | 3            |              | 0            |              |
|    |    | 5          |              | 4            |              | -11          |
| 8  | 8  |            | 7            |              | -11          |              |
|    |    | 12         |              | -7           |              |              |
| 10 | 20 |            | 0            |              |              |              |
|    |    | 12         |              |              |              |              |
| 12 | 32 |            |              |              |              |              |

$$u = \frac{x - x_0}{h} = \frac{x - 6}{2}$$

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By Bessel's Formula

$$y = \frac{1}{2} (y_0 + y_1) + (u - \frac{1}{2}) \Delta y_0 + \frac{1}{2!} u(u-1) \left( \frac{\Delta^2 y_0 + \Delta^2 y_1}{2} \right)$$

$$+ \frac{1}{3!} u(u-1)(u-\frac{1}{2}) \Delta^3 y_0 + \frac{1}{4!} u(u-1)(u+1)(u-2) \left( \frac{\Delta^4 y_0 + \Delta^4 y_1}{2} \right)$$

$$+ \frac{1}{5!} u(u-1)(u+1)(u-2)(u-\frac{1}{2}) (\Delta^5 y_0) + \dots$$

using values

$$y = \frac{1}{2} (3+8) + (u-\frac{1}{2})(5) + \frac{1}{2!} u(u-1) \left( \frac{3+7}{2} \right) +$$

$$\frac{1}{3!} u(u-1)(u-\frac{1}{2})(4) + \frac{1}{4!} u(u-1)(u+1)(u-2) \left( \frac{0-11}{2} \right) +$$

$$\frac{1}{5!} u(u-1)(u+1)(u-2)(u-\frac{1}{2})(-11)$$

$$= \frac{11}{2} + \frac{5u}{2} - \frac{5}{2} + \frac{5u^2}{2} - \frac{5u}{2} + \frac{2}{3} \left( \frac{u^3 - 3u^2 + 1u}{2} \right)$$

$$- \frac{11}{48} (u^4 - 2u^3 - u^2 + 2u) - \frac{11}{120} \left( \frac{u^5 - 5u^4 + 5u^2 - u}{2} \right)$$

$$= \frac{11}{2} + \frac{5u}{2} - \frac{5}{2} + \frac{5u^2}{2} - \frac{5u}{2} + \frac{2}{3} u^3 - \frac{u^2}{3} + \frac{1}{3} u -$$

$$\frac{11}{48} u^4 + \frac{11}{24} u^3 + \frac{11}{48} u^2 - \frac{11}{24} u - \frac{11}{120} u^5 + \frac{11}{48} u^4$$

$$- \frac{11}{48} u^2 + \frac{11}{120} u$$

$$y = -\frac{11}{120} u^5 + \frac{9}{8} u^3 + \frac{3}{2} u^2 + \frac{37}{15} u + 3$$

using value of u.



$$\begin{aligned}
 y &= \frac{-11}{120} \left( \frac{x-6}{2} \right)^5 + \frac{9}{8} \left( \frac{x-6}{2} \right)^3 + \frac{3}{2} \left( \frac{x-6}{2} \right)^2 + \\
 &\quad \frac{37}{15} \left( \frac{x-6}{2} \right) + 3 \\
 &= \frac{-11}{3840} [x^5 - 30x^4 + 36x^3 - 216x^2 + 6480x - 7776] \\
 &\quad + \frac{9}{64} [x^3 - 18x^2 + 108x - 216] + \frac{3}{8} (x^2 - 12x + 36) \\
 &\quad + \frac{37}{30} (x-6) + 3 \\
 &= \frac{-11}{3840} x^5 + \frac{11}{128} x^4 - \frac{33}{320} x^3 + \frac{99}{16} x^2 - \frac{297}{16} x \\
 &\quad + \frac{891}{40} + \frac{9}{64} x^3 - \frac{81}{32} x^2 + \frac{243}{16} x - \frac{243}{8} \\
 &\quad + \frac{3}{8} x^2 - \frac{9}{2} x + \frac{27}{2} + \frac{37}{30} x - \frac{37}{5} + 3 \\
 y = f(x) &= \frac{-11}{3840} x^5 + \frac{11}{128} x^4 - \frac{57}{64} x^3 + \frac{129}{32} x^2 \\
 &\quad - \frac{797}{120} x + 1
 \end{aligned}$$

which is required polynomial.

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 Near Moon CHG Rwp  
 Ph: 4421648, 0333-6126581

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 Asghar Mall Rawalpindi  
 Hello: 051-4410622 - 4411983

# (V) LAPLACE - EVERETT FORMULA :

$$f(x) = \left[ \binom{u}{1} y_1 + \binom{u+1}{3} \Delta^2 y_0 + \binom{u+2}{5} \Delta^4 y_{-1} + \dots \right]$$

$$\left[ \binom{u-1}{1} y_0 + \binom{u}{3} \Delta^2 y_{-1} + \binom{u+1}{5} \Delta^4 y_{-2} + \dots \right]$$

where  $u = x - x_0$

$\binom{u}{i}$  denotes  $\frac{u!}{i!(u-i)!}$

Proof:-

Gauss's Forward formula is

$$\begin{aligned}
 y_x &= y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u+1)}{3!} \Delta^3 y_{-1} \\
 &\quad + \frac{u(u-1)(u+1)(u-2)}{4!} \Delta^4 y_{-2} + \dots
 \end{aligned}$$

It can be written as

$$y_x = y_0 + {}^u C_1 \Delta y_0 + {}^u C_2 \Delta^2 y_{-1} + {}^{u+1} C_3 \Delta^3 y_{-1} + {}^{u+1} C_4 \Delta^4 y_{-2} + {}^{u+2} C_5 \Delta^5 y_{-2} + \dots \quad \text{--- (1)}$$

Now

$$\begin{aligned}
 \Delta^{2k+1} y_{-k} &= \Delta^{2k} (\Delta y_{-k}) \\
 &= \Delta^{2k} (y_{-k+1} - y_{-k})
 \end{aligned}$$

put  $k=0, 1, 2, \dots$  in

$$\Delta^{2k+1} y_{-k} = \Delta^{2k} y_{-k+1} - \Delta^{2k} y_{-k}$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$$

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and so on.  
Put value in ③

$$y_x = y_0 + {}^u C_1 (y_1 - y_0) + {}^u C_2 (\Delta^2 y_0) + {}^{u+1} C_3 (\Delta^2 y_0 - \Delta^2 y_1) \\ + {}^{u+1} C_4 \Delta^4 y_0 + {}^{u+2} C_5 (\Delta^4 y_0 - \Delta^4 y_1) + \dots$$

$$y_x = {}^{u-1} C_0 y_0 + {}^u C_1 y_1 - {}^u C_1 y_0 - ({}^{u+1} C_3 - {}^u C_3) \Delta^2 y_0 \\ + {}^{u+1} C_3 \Delta^2 y_0 - ({}^{u+2} C_5 - {}^{u+1} C_5) \Delta^4 y_0 + {}^{u+2} C_5 \Delta^4 y_0 + \dots$$

$$y_x = -({}^u C_1 - {}^{u-1} C_0) y_0 + {}^u C_1 y_1 - ({}^{u+1} C_3 - {}^u C_3) \Delta^2 y_0 \\ + {}^{u+1} C_3 \Delta^2 y_0 + ({}^{u+2} C_5 - {}^{u+1} C_5) \Delta^4 y_0 + {}^{u+2} C_5 \Delta^4 y_0 + \dots$$

As we know that

$${}^n C_r + {}^n C_{r+1} = {}^{n+1} C_{r+1} \\ \Rightarrow {}^{n+1} C_{r+1} - {}^n C_r = {}^n C_{r+1}$$

By using this identity

$$y_x = -{}^{u-1} C_1 y_0 + {}^u C_1 y_1 - {}^u C_3 \Delta^2 y_0 + {}^{u+1} C_3 \Delta^2 y_0 \\ - {}^{u+1} C_3 \Delta^2 y_0 + {}^{u+2} C_5 \Delta^4 y_0 - \dots$$

$$y_x = [{}^u C_1 y_1 + {}^{u+1} C_3 \Delta^2 y_0 + {}^{u+2} C_5 \Delta^4 y_0 + \dots] - \\ [{}^{u-1} C_1 y_0 + {}^u C_3 \Delta^2 y_0 + {}^{u+1} C_5 \Delta^4 y_0 + \dots] \\ \text{(Assumed Formula)}$$

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\* Question :-

Apply Everett-formula to obtain value of  $y(25)$  Given  $y(20) = 2854$ ,  $y(24) = 3162$ ,  $y(28) = 3544$ ,  $y(32) = 3992$ .

Solution :-

The difference table is

| x        | y    | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|----------|------|------------|--------------|--------------|
| 20       | 2854 |            |              |              |
|          |      | 308        |              |              |
| $x_0$ 24 | 3162 |            | 74           |              |
|          |      | 382        |              | -8           |
| $x_1$ 28 | 3544 |            | 66           |              |
|          |      | 448        |              |              |
| 32       | 3992 |            |              |              |

By Laplace Everett - Formula

$$y_x = [{}^u C_1 y_1 + {}^{u+1} C_3 \Delta^2 y_0 + {}^{u+2} C_5 \Delta^4 y_{-1} + \dots] - \\ [{}^{u-1} C_1 y_0 + {}^u C_3 \Delta^2 y_{-1} + {}^{u+1} C_5 \Delta^4 y_{-2} + \dots]$$

$$u = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

$$= 4 y_1 + \frac{4(u+1)(u-1)}{3!} \Delta^2 y_0 - \frac{u-1}{1} y_0 -$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^2 y_{-1}$$

$$y_{25} = 25(3544) + \frac{(0.25+1)(0.25)(0.25-1)}{3!} (66) -$$



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$$\frac{(0.25-1)(3162) - (0.25)(0.25-1)(0.25-2)(74)}{6}$$

$$= 88.6 - 2.5781 + 2371.5 - 4.047$$

$$y_{1.5} = 3250.9$$

(Answer)

\* Question:-

Apply Everett to following data to find polynomial which satisfies value of  $y$  corresponding to each argument.

|     |   |    |   |     |     |      |
|-----|---|----|---|-----|-----|------|
| $x$ | 0 | 1  | 2 | 3   | 4   | 5    |
| $y$ | 0 | -1 | 8 | 135 | 704 | 2375 |

Solution:- The difference table is

| $x$       | $y$  | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|-----------|------|----------|------------|------------|------------|------------|
| 0         | 0    |          |            |            |            |            |
|           |      | -1       |            |            |            |            |
| 1         | -1   |          | 10         |            |            |            |
|           |      | 9        |            | 108        |            |            |
| $x_0 = 2$ | 8    |          | 118        |            | 216        |            |
|           |      | 127      |            | 324        |            | 120        |
| $x_1 = 3$ | 135  |          | 442        |            | 336        |            |
|           |      | 569      |            | 660        |            |            |
| 4         | 704  |          | 1102       |            |            |            |
|           |      | 1671     |            |            |            |            |
| 5         | 2375 |          |            |            |            |            |

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$$\text{Take } u = \frac{x-x_0}{h} = \frac{x-2}{1} = x-2$$

$$y = \left[ {}^u C_1 y_1 + {}^{u+1} C_3 \Delta^2 y_0 + {}^{u+2} C_5 \Delta^4 y_{-1} + \dots \right] - \left[ {}^{u-1} C_1 y_0 + {}^u C_3 \Delta^2 y_{-1} + {}^{u+1} C_5 \Delta^4 y_{-2} + \dots \right]$$

Using values.

$$y = u(135) + \frac{u(u+1)(u-1)(442)}{3!} + \frac{1}{5!} (u+2)(u+1)u x$$

$$(u-1)(u-2)(336) - (u-1)8 - \frac{u}{3!} (u-1)(u-2)(118)$$

$$- \frac{1}{5!} (u+1)(u)(u-1)(u-2)(u-3)(216)$$

$$= 135u + \frac{221(u^3-u)}{3} + \frac{14(u^5-5u^3+4u)-8u}{5}$$

$$+ \frac{8-59(u^3-3u^2+2u)}{3} - \frac{9(u^5-5u^4+5u^3+5u^2-6u)}{5}$$

$$= u^5 + 9u^4 + 31u^3 + 50u^2 + 36u + 8$$

$$f(x) = u^5 + 9u^4 + 31u^3 + 50u^2 + 36u + 8$$

$$\text{put } u = x-2$$

$$f(x) = (x-2)^5 + 9(x-2)^4 + 31(x-2)^3 + 50(x-2)^2 + 36(x-2) + 8$$

$$f(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

$$+ 9(x^4 - 8x^3 + 24x^2 - 32 + 16) +$$

$$31(x^3 - 6x^2 + 12x - 8) + 50(x^2 + 44x)$$

$$+ 36x - 72 + 8$$



$$f(x) = x^5 + (-10+9)x^4 + (40-72+31)x^3 + (-80+50+216-186)x^2 + (80-288+372+36-200)x + (-32+144-248-72+8+200)$$

$$f(x) = x^5 - x^4 - x^3$$

which is required  
polynomial.

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FOR

## UNEQUALLY SPACED DATA

(I) AITKEN'S METHOD :- First the  
table for Aitken's Method is given by.

| $x_k$ | $y_k$ |              |                |                  | $x_i - x$ |
|-------|-------|--------------|----------------|------------------|-----------|
| $x_0$ | $y_0$ |              |                |                  | $x_0 - x$ |
| $x_1$ | $y_1$ | $p_{0,1}(x)$ |                |                  | $x_1 - x$ |
| $x_2$ | $y_2$ | $p_{0,2}(x)$ | $p_{0,1,2}(x)$ |                  | $x_2 - x$ |
| $x_3$ | $y_3$ | $p_{0,3}(x)$ | $p_{0,1,3}(x)$ | $p_{0,1,2,3}(x)$ | $x_3 - x$ |

where

$$p_{0,k}(x) = \frac{1}{x_k - x_0}$$

$$y_0 \quad x_0 - x$$

$$y_k \quad x_k - x$$

$$p_{0,1,k}(x) = \frac{1}{x_k - x_1}$$

$$p_{0,1}(x) \quad x_1 - x$$

$$p_{0,k}(x) \quad x_k - x$$

$$p_{0,1,2,k}(x) = \frac{1}{x_k - x_2}$$

$$p_{0,1,2}(x) \quad x_2 - x$$

$$p_{0,1,k}(x) \quad x_k - x$$

(In above table  $p_{0,1,2,3}(x)$  is required  
answer.)

Question:-

Apply Aitken's iteration to following data to find  $f(2)$ .

|        |   |    |   |    |
|--------|---|----|---|----|
| $x$    | 0 | 1  | 4 | 6  |
| $f(x)$ | 1 | -1 | 1 | -1 |

Solution:

| $x$ | $y$ | $p_{0,k}(x)$                                 | $p_{0,1,k}(x)$  | $p_{0,1,2,k}(x)$                         |
|-----|-----|--|---|--|
| 0   | 1   |  |   |  |
| 1   | -1  | $\frac{1(-1) - 1(-2)}{-1 - (-2)} = 3$        |   |  |
| 4   | 1   | $\frac{1(2) - 1(-2)}{2 - (-2)} = 1$          | $\frac{-3(2) - 1(1)}{2 - (-1)} = -\frac{5}{3}$            |  |
| 6   | -1  | $\frac{1(4) - 1(2)}{4 - (-2)} = \frac{1}{3}$ | $\frac{-3(4) - \frac{1}{3}(-1)}{4 - (-1)} = -\frac{7}{3}$ | $\frac{-5/3(4) - (-7/3)(2)}{4 - 2} = -1$ |

$$\text{Where } x_0 - x = 0 - 2 = -2$$

$$x_1 - x = 1 - 2 = -1$$

$$x_2 - x = 4 - 2 = 2$$

$$x_3 - x = 6 - 2 = 4$$

$$y(2) = f(2) = -1$$

(Answer)

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Remark:-

$$x_1 - x_0 = (x_1 - x) - (x_0 - x)$$

(From previous table)

\* Question:-

Apply Aitken's Method Find  $f(3)$ .

|     |   |   |   |   |
|-----|---|---|---|---|
| $x$ | 0 | 1 | 2 | 4 |
| $y$ | 1 | 1 | 2 | 5 |

Solution:-  $x = 3$ 

| $x$ | $y$ | $p_{0,k}(x)$   | $p_{0,1,k}(x)$   | $p_{0,1,2,k}(x)$  | $x_{k-3}$ |
|-----|-----|--|--|---|-----------|
| 0   | 1   |  |  |   | -3        |
| 1   | 1   | $\frac{1}{1-0} \left  \begin{array}{c} 1 \\ 1 \end{array} \right  \begin{array}{c} -3 \\ -2 \end{array} = 1$   |  |   | -2        |
| 2   | 2   | $\frac{1}{2-0} \left  \begin{array}{c} 1 \\ 2 \end{array} \right  \begin{array}{c} -3 \\ -1 \end{array} = 5/2$ | $\frac{1}{5/2-1} \left  \begin{array}{c} 1 \\ 5/2 \end{array} \right  \begin{array}{c} -2 \\ -1 \end{array} = 4$ |   | -1        |
| 4   | 5   | $\frac{1}{4-0} \left  \begin{array}{c} 1 \\ 4 \end{array} \right  \begin{array}{c} -3 \\ 1 \end{array} = 4$    | $\frac{1}{4-1} \left  \begin{array}{c} 1 \\ 4 \end{array} \right  \begin{array}{c} -2 \\ 1 \end{array} = 3$      | $\frac{1}{(4-2)} \left  \begin{array}{c} 4 \\ 3 \end{array} \right  \begin{array}{c} -1 \\ 1 \end{array} = \frac{7}{2}$ | 1         |

$$\Rightarrow y(3) = \frac{7}{2} \text{ is required Answer.}$$

\* Question:-

Use Aitken's iteration to following data to find  $f(3)$ .

|     |   |    |    |    |   |
|-----|---|----|----|----|---|
| $x$ | 0 | 1  | 2  | 4  | 5 |
| $y$ | 0 | 16 | 48 | 88 | 0 |

Solution:-

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| $x$ | $y$ | $P_{0,k}(x)$   | $P_{0,l,k}(x)$  | $P_{0,i,l,k}(x)$                                       | $P_{0,i,l,k}(x)$ |
|-----|-----|--|---|--|------------------|
| 0   | 0   |  |   |  |                  |
| 1   | 16  | $\begin{vmatrix} 0 & -3 \\ 16 & -2 \end{vmatrix} = 48$ | $\begin{vmatrix} 48 & -2 \\ 72 & -1 \end{vmatrix} = 96$ |  |                  |
| 2   | 48  | $\begin{vmatrix} 0 & -3 \\ 48 & -1 \end{vmatrix} = 72$ | $\begin{vmatrix} 48 & -2 \\ 66 & 1 \end{vmatrix} = 60$  | $\begin{vmatrix} 96 & -1 \\ 60 & 1 \end{vmatrix} = 78$ |                  |
| 4   | 88  | $\begin{vmatrix} 0 & -3 \\ 88 & 1 \end{vmatrix} = 16$  | $\begin{vmatrix} 48 & -2 \\ 0 & 2 \end{vmatrix} = 24$   | $\begin{vmatrix} 96 & -1 \\ 24 & 2 \end{vmatrix} = 72$ |                  |
| 5   | 0   | $\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} = 0$    |   |  |                  |

## (II) LAGRANGE'S INTERPOLATION

### FORMULA

Let  $f(x_0), f(x_1), \dots, f(x_n)$  be values of function  $y = f(x)$  corresponding to arguments  $x_0, x_1, x_2, \dots, x_n$  not necessarily equally spaced, then...

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) +$$

$$\dots +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n).$$

Proof:- Suppose  $f(x)$  is a polynomial of degree  $n$ .

Let

$$f(x) = (x-x_1)(x-x_2)(x-x_3)\dots(x-x_n) A_0 +$$

$$(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n) A_1 +$$

$$(x-x_0)(x-x_1)(x-x_3)(x-x_4)\dots(x-x_n) A_2 +$$

$$\dots +$$

$$(x-x_0)(x-x_1)(x-x_3)\dots(x-x_{n-1}) A_n \rightarrow (i)$$

where  $A$ 's are constant to be determined.

Put  $x = x_0$



$$f(x_0) = \frac{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n) A_0}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}$$

Similarly; Put  $x = x_1$  we get

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}$$

Continuing in this way

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

Put all  $A$ 's value in (i), we get

$$f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} f(x_1) +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3) \dots (x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \dots (x_2 - x_n)} f(x_2) +$$

$$\dots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} f(x_n)$$

Which is Lagrange's Interpolation  
Formula.

\* Question:-

Find polynomial function  $f(x)$  given that  $f(0)=2$ ,  $f(1)=3$ ,  $f(2)=12$  and  $f(3)=35$ . Also find  $f(5)$ .

Solution:-

|        |   |   |    |    |
|--------|---|---|----|----|
| $x$    | 0 | 1 | 2  | 3  |
| $f(x)$ | 2 | 3 | 12 | 35 |

By Lagrange's Formula:-

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} (2) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} (3)$$

$$+ \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} (12) + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} (35)$$

$$= \frac{x^3 - 6x^2 + 11x - 6}{-3} + \frac{3(x^3 - 5x^2 + 6x)}{2}$$

$$- 6(x^3 - 4x^2 + 3x) + \frac{35(x^3 - 3x^2 + 2x)}{6}$$

$$f(x) = x^3 + x^2 - x + 2$$

$$f(5) = 5^3 + 5^2 - 5 + 2$$

$$= 125 + 25 - 5 + 2$$

$$= 147$$

Required polynomial

(Answer)

\* Question:-

The value of  $x$  &  $y$  are given as  
below  $x: 5 \quad 6 \quad 9 \quad 11$   
 $y: 12 \quad 13 \quad 14 \quad 16$

Find value of  $y$  when  $x=10$ . using  
Lagrange's Formula for unequal interval.

Solution:- Given

|     |    |    |    |    |
|-----|----|----|----|----|
| $x$ | 5  | 6  | 9  | 11 |
| $y$ | 12 | 13 | 14 | 16 |

The Lagrange's Formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

At  $x=10$  using values we have

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} (12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} (13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} (14) +$$

$$\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} (16)$$

$$f(10) = \frac{-4}{-24} (12) - \frac{5}{+15} (13) - \frac{20}{-24} (14) + \frac{20}{60} (16)$$

$$f(10) = 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$f(10) = 14.667$$

(Answer).

\* Question:-

Given  $\log_{10} 654 = 2.8156$ ,  $\log_{10} 658 = 2.8182$

$\log_{10} 659 = 2.8189$ ,  $\log_{10} 661 = 2.8202$ .

Find  $\log_{10} 656$ .

Solution:- Let  $f(x) = \log_{10}(x)$

|        |        |        |        |        |
|--------|--------|--------|--------|--------|
| $x$    | 654    | 658    | 659    | 661    |
| $f(x)$ | 2.8156 | 2.8182 | 2.8189 | 2.8202 |

Applying  
Lagrange's Formula

(2.8156)

$$\log_{10}(656) = f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} (2.8182) +$$

$$\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} (2.8189) +$$

$$\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} (2.8202)$$



$$f(656) = \frac{-30}{-140} (2.8156) + \frac{30}{12} (2.8182) + \frac{20}{-10} (2.8189) + \frac{12}{42} (2.8202)$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$

$$f(656) = 2.8168$$

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(Answer)

### \* THEOREM:-

Let  $L_i(x)$  denotes the Lagrange's polynomial of degree less than or equal to  $n$  and

$$\prod_{i=0}^n (x - x_i) = (x - x_0)(x - x_1) \dots (x - x_n)$$

$$\prod_{i=0}^n (x_i - x_j) = (x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$$

then

$$L_i(x) = \frac{\prod_{j=0, j \neq i}^n (x - x_j)}{(x - x_i) \prod_{j=0, j \neq i}^n (x_i - x_j)} \quad i = 0, 1, 2, \dots, n$$

Proof:-

From Lagrange's interpolation formula

$$L_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1}) \dots (x_i - x_n)}$$

ring and ÷ing by  $(x - x_i)$  we have

$$L_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x - x_i)(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1}) \dots (x_i - x_n)}$$

$$= \frac{\prod_{j=0}^n (x - x_j)}{(x - x_i) \prod_{j=0}^n (x_i - x_j)} \quad \text{proved.}$$

Theorem:-

With the meaning of symbol as in previous theorem, prove that

$$\sum_{i=0}^n \frac{\prod_{j=0}^n (x - x_j)}{(x - x_i) \prod_{j=0}^n (x_i - x_j)} = 1 \quad \text{i.e.} \quad \sum_{i=0}^n L_i(x) = 1$$

Proof:- We know that

$$L_i(x) = \frac{\prod_{j=0}^n (x - x_j)}{(x - x_i) \prod_{j=0}^n (x_i - x_j)} \rightarrow (1)$$

$$\text{Now } \prod_{i=0}^n (x - x_i) = (x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)$$

$$\Rightarrow \frac{1}{\prod_{i=0}^n (x - x_i)} = \frac{1}{(x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)}$$

$$= \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + \dots + \frac{A_i}{x - x_i} + \dots + \frac{A_n}{x - x_n} \rightarrow (2)$$

$$\Rightarrow \frac{1}{\prod_{i=0}^n (x - x_i)} = \sum_{i=0}^n \frac{A_i}{(x - x_i)} \rightarrow (3)$$



ing (2) by  $x - x_i$  on both sides

$$\frac{(x-x_i)}{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)} = \frac{A_0(x-x_i)}{(x-x_0)} + \frac{A_1(x-x_i)}{x-x_1} + \dots + \frac{A_i(x-x_i)}{x-x_i} + \dots + \frac{A_n(x-x_i)}{x-x_n}$$

Putting  $x = x_i$

$$\frac{1}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} = A_i$$

$$\Rightarrow A_i = \frac{1}{\prod_{j=0, j \neq i}^n (x_i - x_j)}$$

Put in (3)

$$\frac{1}{\prod_{i=0}^n (x - x_i)} = \sum_{i=0}^n \frac{1}{(x - x_i) \prod_{j=0, j \neq i}^n (x_i - x_j)}$$

$$\Rightarrow \sum_{i=0}^n \frac{\prod_{j=0, j \neq i}^n (x - x_j)}{(x - x_i) \prod_{j=0, j \neq i}^n (x_i - x_j)} = 1$$

From (i)

$$\sum_{i=0}^n L_i(x) = 1$$

(proved)

\* Inverse Lagrange's Interpolation Formula:-  
If for a function  $f$ , has values  $f(x_0), f(x_1), \dots, f(x_n)$  at  $x_0, x_1, \dots, x_n$  then for given  $f(x)$

$$x = \frac{[f(x)-f(x_1)][f(x)-f(x_2)]\dots[f(x)-f(x_n)]}{[f(x_0)-f(x_1)][f(x_0)-f(x_2)]\dots[f(x_0)-f(x_n)]} x_0$$

$$+ \frac{[f(x)-f(x_0)][f(x)-f(x_2)]\dots[f(x)-f(x_n)]}{[f(x_1)-f(x_0)][f(x_1)-f(x_2)]\dots[f(x_1)-f(x_n)]} x_1$$

$$+ \dots + \frac{[f(x)-f(x_0)][f(x)-f(x_1)]\dots[f(x)-f(x_{n-1})]}{[f(x_n)-f(x_0)][f(x_n)-f(x_1)]\dots[f(x_n)-f(x_{n-1})]} x_n$$

Example:-

Given

|        |   |   |    |    |
|--------|---|---|----|----|
| $x$    | 1 | 3 | 4  | 5  |
| $f(x)$ | 6 | 8 | 15 | 20 |

and  $f(x) = 18$  Find  $x$ .

Solution:-

Given that

|        |   |   |    |    |
|--------|---|---|----|----|
| $x$    | 1 | 3 | 4  | 5  |
| $f(x)$ | 6 | 8 | 15 | 20 |

By using inverse Lagrange's formula:

$$x = \frac{(18-8)(18-15)(18-20)(1)}{(6-8)(6-15)(6-20)} + \frac{(18-6)(18-15)(18-20)(3)}{(8-6)(8-15)(8-20)} + \frac{(18-6)(18-8)(18-20)(4)}{(15-6)(15-8)(15-20)} + \frac{(18-6)(18-8)(18-15)(5)}{(20-6)(20-8)(20-15)}$$

$$x = \frac{(10)(3)(-2)(1)}{(-2)(-9)(-14)} + \frac{(12)(3)(-2)(3)}{(2)(-7)(-12)} + \frac{(12)(10)(-2)(4)}{(9)(7)(-5)} + \frac{(12)(10)(3)(5)}{(14)(12)(5)}$$

$$= 0.2381 - 1.2857 + 3.0476 + 2.1429$$

$$x = 4.6429$$

(Answer)

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### (III) NEWTON'S DIVIDED DIFFERENCE FORMULA

\* Divided Difference:-

Let  $x_0, x_1, x_2, \dots, x_n$  be the arguments, which are not necessarily equally spaced and a function  $f$  has the values  $f(x_0), f(x_1), \dots, f(x_n)$  over these arguments respectively. Then the quantities

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\dots \dots f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

are called 1<sup>st</sup> divided difference

And

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \text{ etc}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \text{ etc}$$

$$\vdots$$

$$f[x_0, x_1, \dots, x_{n-1}, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} \text{ etc}$$

are respectively called 2<sup>nd</sup>, 3<sup>rd</sup>, ..., n<sup>th</sup> divided differences.



f. THEOREM:-

Prove the symmetric property of divided difference.  
(OR)

Show that divided difference is independent of order of argument.

Proof:- For 1<sup>st</sup> order:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This can be written as

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_0)}{x_1 - x_0} \\ &= \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \\ &= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \end{aligned}$$

$$\Rightarrow f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \quad \rightarrow (*)$$

Now

$$\begin{aligned} f[x_1, x_0] &= \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \quad (\because \text{by } * \text{ def}) \\ &= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \\ &= f[x_0, x_1] \end{aligned}$$

$$\Rightarrow f[x_0, x_1] = f[x_1, x_0]$$

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For 2<sup>nd</sup> Order:

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{1}{x_2 - x_0} \left[ \frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_1 - x_0} \right] \quad (\text{by } *) \end{aligned}$$

$$= \frac{1}{x_2 - x_0} \left[ \frac{f(x_1)}{x_1 - x_2} \left( \frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0} \right) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} \right]$$

$$= \frac{1}{x_2 - x_0} \left[ \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_0)} + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} \right]$$

$$= \frac{1}{x_2 - x_0} \left[ \frac{f(x_1)(x_2 - x_0)}{(x_1 - x_2)(x_1 - x_0)} + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} \right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_0 - x_1)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \rightarrow (A)$$

Now by using above concept we write  $f[x_1, x_2, x_0]$  as

$$\begin{aligned} f[x_1, x_2, x_0] &= \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \\ &\quad + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} \\ &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

$\rightarrow (B)$

By (A) and (B)

$$f[x_1, x_2, x_0] = f[x_0, x_1, x_2]$$



$$1^{\text{st}} \text{ Divided Difference} = \frac{\Delta}{h} = \frac{\Delta}{1!h}$$

$$2^{\text{nd}} \text{ divided difference} = \frac{\Delta^2}{2h^2} = \frac{\Delta^2}{2!h^2}$$

$$3^{\text{rd}} \text{ divided difference} = \frac{\Delta^3}{6h^3} = \frac{\Delta^3}{3!h^3}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n^{\text{th}} \text{ divided difference} = \frac{\Delta^n}{n!h^n}$$

Hence  $n^{\text{th}}$  divided difference of  $f(x)$  is

$$f(x) = \frac{1}{n!h^n} [\text{n}^{\text{th}} \text{ forward difference of } f(x)]$$

$$= \frac{1}{n!h^n} [\text{constant}]$$

$$= \text{constant.}$$

And  $(n+1)^{\text{th}}$  divided difference of  $f(x)$  is

$$f(x) = \frac{1}{(n+1)!h^{(n+1)}} [(n+1)^{\text{th}} \text{ forward diff. of } f(x)]$$

$$= \frac{1}{(n+1)!h^{(n+1)}} (0)$$

$$= 0$$

Hence it is proved that  $n^{\text{th}}$  divided difference of polynomial of degree  $n$  is constant all all higher differences are zero.

(proved).

## NEWTON'S Divided DIFFERENCE FORMULA

Let a function  $f$  has values  $f(x_0), f(x_1), \dots, f(x_n)$  at argument  $x_0, x_1, \dots, x_n$  which are not necessarily equally spaced then

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + \dots +$$

$$(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})f[x_0, x_1, \dots, x_n]$$

Proof:- Consider  $f[x, x_0] = \frac{f(x_0) - f(x)}{x_0 - x}$

$$\Rightarrow (x_0 - x)f[x, x_0] = f(x_0) - f(x)$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0)f[x, x_0] \rightarrow (i)$$

Next

$$f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$$\Rightarrow (x_1 - x)f[x, x_0, x_1] = f[x_0, x_1] - f[x, x_0]$$

$$\Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]$$

by  $f[x, x_0]$  value in (i),

$$f(x) = f(x_0) + (x - x_0)[f[x_0, x_1] + (x - x_1)f[x_0, x_1, x_2]]$$

$$= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x, x_0, x_1] \rightarrow (ii)$$

Also

$$f[x, x_0, x_1, x_2] = \frac{f[x_0, x_1, x_2] - f[x, x_0, x_1]}{x_2 - x}$$

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$$\Rightarrow f[x, x_0, x_1] = f[x_0, x_1, x_2] + \frac{(x-x_2)f[x, x_0, x_1, x_2]}{(x-x_2)}$$

putting in (ii)

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})f[x, x_0, x_1, \dots, x_n]$$

Continuing this way we obtain.

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})f[x, x_0, x_1, \dots, x_n]$$

Now as  $n^{\text{th}}$  divided difference of a polynomial of degree  $n$  is constant and all higher differences are zero, so being  $(n+1)^{\text{st}}$  divided difference  $f[x, x_0, x_1, \dots, x_n] = 0$

Thus finally;

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0, x_1, x_2, x_3, x_4] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})f[x_0, x_1, \dots, x_{n-1}]$$

(proved).

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\* Question:-

The observed values of function are respectively 168, 120, 72 & 63 at four positions 3, 7, 9 and 10 of indep. Variable. what is best estimate you can give for value of function at position 6 of indep. Variable.

Solution:- Divided Difference Table is

| $x$ | $f(x)$ | $1^{\text{st}} \text{ D.D}$ | $2^{\text{nd}} \text{ D.D}$ | $3^{\text{rd}} \text{ D.D}$ |
|-----|--------|-----------------------------|-----------------------------|-----------------------------|
| 3   | 168    |                             |                             |                             |
|     |        | $\frac{120-168}{7-3} = -12$ |                             |                             |
| 7   | 120    |                             | $\frac{-24+12}{9-3} = -2$   |                             |
|     |        | $\frac{72-120}{9-7} = -24$  |                             | $\frac{5+2}{10-3} = 1$      |
| 9   | 72     |                             | $\frac{-9+24}{10-7} = 5$    |                             |
|     |        | $\frac{63-72}{10-9} = -9$   |                             |                             |
| 10  | 63     |                             |                             |                             |

Newton's d.d. Formula

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \dots$$

As  $x = 6$ ,  $f(x_0) = 168$

$$\begin{aligned} f(6) &= 168 + (6-3)(-12) + (6-3)(6-7)(-2) \\ &\quad + (6-3)(6-7)(6-9)(1) \\ &= 168 - 36 + 6 + 9 \\ &= 147 \end{aligned}$$

(Answer)



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\* Question:-

Find polynomial satisfying data by Newton's divided difference formula

 $x: 0 \quad 1 \quad 2 \quad 3$  $f(x): 1 \quad 2 \quad 11 \quad 34$ 

Solution:- Divided Difference Table

| $x$ | $f(x)$ | $1^{st} D.D$             | $2^{nd} D.D$           | $3^{rd} D.D$          |
|-----|--------|--------------------------|------------------------|-----------------------|
| 0   | 1      | $\frac{2-1}{1-0} = 1$    |                        |                       |
| 1   | 2      | $\frac{11-2}{2-1} = 9$   | $\frac{9-1}{2-0} = 4$  |                       |
| 2   | 11     | $\frac{34-11}{3-2} = 23$ | $\frac{23-9}{3-1} = 7$ | $\frac{7-4}{3-0} = 1$ |
| 3   | 34     |                          |                        |                       |

Newton's D.D Formula is

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \dots$$

$$\begin{aligned} f(x) &= 1 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)(1) \\ &= 1 + x + 4x^2 - 4x + x(x^2 - 3x + 2) \\ &= 1 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x \\ &= x^3 + x^2 - x + 1 \end{aligned}$$

(Answer).

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Note:-

In this question, arguments are equally spaced. Hence we can find  $f(x)$  using Newton's forward or backward formula.

\* Question:- Apply Newton's D.D Formula to find no. of person getting 6 rupees from the following data.

|                     |     |     |     |     |    |
|---------------------|-----|-----|-----|-----|----|
| Incom per day (Rs): | 3   | 5   | 7   | 8   | 10 |
| No. of person       | 180 | 154 | 120 | 110 | 90 |

Solution:- Divided difference Table is

| $x$ | $y$ | $1^{st} D.D$                | $2^{nd} D.D$                | $3^{rd} D.D$                  | $4^{th} D.D$                      |
|-----|-----|-----------------------------|-----------------------------|-------------------------------|-----------------------------------|
| 3   | 180 | $\frac{154-180}{5-3} = -13$ |                             |                               |                                   |
| 5   | 154 | $\frac{120-154}{7-5} = -17$ | $\frac{-17+13}{7-3} = -1$   |                               |                                   |
| 7   | 120 | $\frac{110-120}{8-7} = -10$ | $\frac{-10+17}{8-5} = 2.33$ | $\frac{-2+33+1}{8-3} = 0.67$  |                                   |
| 8   | 110 | $\frac{90-110}{10-8} = -10$ | $\frac{-10+10}{10-7} = 0$   | $\frac{0+2.33}{10-5} = -0.47$ | $\frac{-0.47-0.67}{10-3} = -0.16$ |
| 10  | 90  |                             |                             |                               |                                   |

Using Newton's Divided Difference formula

given that  $x = 6$ 

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$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1) \cdot f[x_0, x_1, x_2] + \dots$$

$$f(6) = 180 + (6-3)(-13) + (6-3)(6-5)(-1) + (6-3)(6-5)(6-7)(0.67) + (6-3)(6-5)(6-7)(6-8)(-0.16)$$

$$= 135$$

No. of person getting Rs. 6 per day is 135

(Answer).

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## Exercise

(i) Find polynomial satisfying ordered pair  $(-1, 33)$ ,  $(0, 5)$ ,  $(2, 9)$  and  $(5, 1335)$  by N.D.D. formula.

$$\text{Ans: } 3x^4 - 5x^3 + 6x^2 - 11x + 5$$

(ii) Find  $\ln(9.2)$  From

$$\ln(8.80) = 2.0794$$

$$\ln(9.0) = 2.1972$$

$$\ln(9.5) = 2.2513 \text{ by newton D.D. formula.}$$

(iii) The following tables is give. what is form of function

(i)

|     |   |   |    |     |
|-----|---|---|----|-----|
| $x$ | 0 | 1 | 2  | 5   |
| $y$ | 2 | 3 | 12 | 147 |

$$\text{Ans: } x^3 - x + 2$$

(ii)

|        |   |    |    |     |
|--------|---|----|----|-----|
| $x$    | 0 | 1  | 4  | 5   |
| $f(x)$ | 8 | 11 | 78 | 123 |

$$\text{Ans: } \frac{1}{6}(x^3 + 24x^2 - 7x + 48)$$

(iv) Find polynomial of lowest possible degree which assumes values 3, 12, 15, -2 when  $x = 3, 2, 1, -1$  respectively by using newton's divided difference formula.

$$\text{Ans: } x^3 - 9x^2 + 17x + 6$$

# NUMERICAL DIFFERENTIATION

It is the technique of evaluating  $f'(x)$  or  $f''(x)$  or  $f'''(x)$  etc. when for a function  $f(x)$ , it is given that at some arguments  $x_0, x_1, \dots, x_n$  has values  $f(x_0), f(x_1), \dots, f(x_n)$ .

## CASE (1):-

In either of the following cases.

- 1) To calculate  $f'(x)$ ,  $f''(x)$  or  $f'''(x)$  etc when  $x \in [x_0, x_n]$  or  $x \notin [x_0, x_n]$ .
- 2) To calculate  $f'(x)$ ,  $f''(x)$  or  $f'''(x)$  etc when  $x_0, x_1, \dots, x_n$  are equally spaced or  $x_0, x_1, \dots, x_n$  are not equally spaced.
- 3) To calculate  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$  etc when  $x = x_k$  or  $x \neq x_k$  for  $0 \leq k \leq n$ .

Just calculate  $f(x)$  by any appropriate Method (Like Newton's Formulae, Gauss Formulae, Lagrange's Method etc) and then by  $f(x)$  find  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$  etc.

## Example:-

Find  $f'(x)$  and  $f''(x)$  at  $x=1.5$

from table

|        |   |    |    |    |
|--------|---|----|----|----|
| $x$    | 1 | 2  | 3  | 4  |
| $f(x)$ | 4 | 11 | 30 | 67 |

Solution:-

As given

|        |   |    |    |    |
|--------|---|----|----|----|
| $x$    | 1 | 2  | 3  | 4  |
| $f(x)$ | 4 | 11 | 30 | 67 |

By Lagrange Interpolation formula:-

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f(x) = \frac{(x-2)(x-3)(x-4)(4)}{(1-2)(1-3)(1-4)} + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \quad (1)$$

$$+ \frac{(x-1)(x-2)(x-4)(30)}{(3-1)(3-2)(3-4)} + \frac{(x-1)(x-2)(x-3)(67)}{(4-1)(4-2)(4-3)}$$

After Simplification we have



$$f(x) = x^3 + 3$$

$$f'(x) = 3x^2 \text{ and } f''(x) = 6x$$

$$f'(1.5) = 3(1.5)^2 = 6.75 \quad f''(1.5) = 6(1.5) = 9.0$$

(Answer)

2<sup>nd</sup> Method:

Using Newton's Forward formula

| x | f(x) | $\Delta$ | $\Delta^2$ | $\Delta^3$ |
|---|------|----------|------------|------------|
| 1 | 4    |          |            |            |
|   |      | 7        |            |            |
| 2 | 11   |          | 12         |            |
|   |      | 19       |            | 6          |
| 3 | 30   |          | 18         |            |
|   |      | 37       |            |            |
| 4 | 67   |          |            |            |

$$x_0 = 1, h = 1$$

$$u = \frac{x - x_0}{h} = x - 1$$

By Newton's F. Formula

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0)$$

$$= 4 + (x-1)(7) + \frac{(x-1)(x-2)}{2!}(12) +$$

$$\frac{(x-1)(x-2)(x-3)}{3!}(6)$$

$$= x^3 + 3$$

$$f(x) = x^3 + 3$$

$$f'(x) = 3x^2$$

$$f'(1.5) = 3(1.5)^2 = 6.75$$

$$\text{and } f''(x) = 6x$$

$$f''(1.5) = 6(1.5) = 9.0$$

Ans

Note:

If argument are not equally spaced then construct Divided difference table and use divided difference formula.

Case (II):

Numerical Differentiation at a point included in the table of value of function and given arguments are equally spaced. In this case, two (formulae) or techniques are used which are

(i) Numerical Differentiation by forward Difference operator

(ii) Numerical Differentiation by backward Difference operator.



ii. Numerical Differentiation by forward Difference Operator:-

As by shift operator

$$E f(a) = f(a+h) \\ = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$

by Taylor expansion

$$= f(a) + h D f(a) + \frac{h^2}{2!} D^2 f(a) + \frac{h^3}{3!} D^3 f(a) + \dots$$

$$= \left[ 1 + hD + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \dots \right] f(a)$$

$$E f(a) = e^{hD} f(a)$$

$$\Rightarrow E = e^{hD} \rightarrow \text{ii,}$$

But if we consider forward difference operator

$$E = 1 + \Delta \rightarrow \text{iii,}$$

$\Rightarrow$  By ii, and iii,

$$e^{hD} = 1 + \Delta$$

$$hD = \ln(1 + \Delta) \quad \therefore \text{taking log on b.s}$$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots$$

$$D = \frac{1}{h} \left[ \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right]$$

$$D^n = \frac{1}{h^n} \left[ \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right]^n \rightarrow (*)$$

when put  $n=1$  in (\*) we get

$$D = \frac{1}{h} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right)$$

$$f'(a) = \frac{1}{h} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right) f(a) \rightarrow (1)$$

When  $n=2$

$$f''(a) = D^2 f(a) \\ = \frac{1}{h^2} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right)^2 f(a)$$

$$= \frac{1}{h^2} \left( \Delta^2 - \frac{1}{2} \Delta^3 - \frac{1}{2} \Delta^3 + \frac{1}{3} \Delta^4 + \frac{1}{3} \Delta^4 + \frac{1}{4} \Delta^4 - \frac{1}{4} \Delta^5 - \frac{1}{6} \Delta^5 - \frac{1}{6} \Delta^5 + \frac{1}{4} \Delta^6 + \dots \right) f(a)$$

$$= \frac{1}{h^2} \left( \Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 - \frac{5}{6} \Delta^5 + \dots \right) f(a) \rightarrow (2)$$

When  $n=3$

$$f'''(a) = \frac{1}{h^3} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right)^3 f(a)$$

$$= \frac{1}{h^3} \left( \Delta^3 - \frac{3}{2} \Delta^4 + \frac{7}{4} \Delta^5 - \frac{15}{8} \Delta^6 + \dots \right) f(a) \rightarrow (3)$$

(1), (2) and (3) are 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> derivatives of  $f(x)$ .

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Example:-

Given data

X: 0 1 2 3 4 5  
 Y: 3 4 19 84 259 628

Find  $f'(1)$  and  $f''(1)$ .

Solution: The difference table is

| X   | Y   | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|-----|-----|----------|------------|------------|------------|------------|
| 0   | 3   |          |            |            |            |            |
| → 1 | 4   | 15       |            |            |            |            |
|     |     |          | 14         |            |            |            |
| 2   | 19  |          | 50         | 36         |            |            |
|     |     | 65       |            | 60         | 24         |            |
| 3   | 84  |          | 110        |            | 24         | 0          |
|     |     | 175      |            | 84         |            |            |
| 4   | 259 |          | 194        |            |            |            |
|     |     | 369      |            |            |            |            |
| 5   | 628 |          |            |            |            |            |

Here  $a = 1$ 

The first derivative is given by

$$Df(a) = \frac{1}{h} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right) f(a)$$

$$f'(1) = \frac{1}{h} \left( \Delta f(1) - \frac{\Delta^2 f(a)}{2} + \frac{\Delta^3 f(a)}{3} - \frac{\Delta^4 f(a)}{4} + \dots \right)$$

$$= \frac{1}{1} \left[ 15 - \frac{1(50)}{2} + \frac{1(60)}{3} - \frac{1(24)}{4} \right]$$

$$f'(1) = 15 - 25 + 20 - 6$$

$$f'(1) = 4$$

The second derivative is given by

$$D^2 f(a) = \frac{1}{h^2} \left[ \Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 - \frac{5}{6} \Delta^5 + \dots \right] f(a)$$

$$f''(1) = \frac{1}{1} \left[ 50 - 60 + \frac{11}{12} (24) \right]$$

$$= 50 - 60 + 22 = 12$$

$$f''(1) = 22$$

(Answer)

Example:-

Calculate first three derivatives from data

X: 0 1 2 3 4 5  
 Y: 2 2 8 26 62 128

at  $x = 2$ 

Solution:-

| X   | Y   | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|-----|-----|----------|------------|------------|------------|
| 0   | 2   |          |            |            |            |
| 1   | 2   | 0        |            |            |            |
| → 2 | 8   | 6        | 6          |            |            |
|     |     | 18       | 12         | 6          | 0          |
| 3   | 26  |          | 18         | 6          | 0          |
|     |     | 36       |            | 6          |            |
| 4   | 62  |          | 24         |            |            |
|     |     | 60       |            |            |            |
| 5   | 128 |          |            |            |            |

At  $x=2$  first derivative is

$$Df(2) = \frac{1}{h} \left[ \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right] f(2)$$

$$= \frac{1}{1} \left[ 18 - \frac{1}{2} (18) + \frac{1}{3} (6) \right]$$

$$= 18 - 9 + 2$$

$$f'(2) = 11$$

The 2<sup>nd</sup> derivative at  $x=2$

$$f''(2) = \frac{1}{h^2} [\Delta^2 - \Delta^3] f(2)$$

$$= \frac{1}{1} [18 - 6]$$

$$= 12$$

3<sup>rd</sup> derivative at  $x=2$

$$f'''(x) = \frac{1}{h^3} [\Delta^3] f(2)$$

$$= \frac{1}{1} (6)$$

$$f'''(x) = 6$$

Answer

## (ii) Numerical Differentiation by Backward Difference Operator :-

Consider

$$Ef(a) = f(a+h)$$

$$= f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots$$

$$= f(a) + hDf(a) + \frac{h^2}{2!} D^2f(a) + \frac{h^3}{3!} D^3f(a) + \dots$$

$$= \left( 1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots \right) f(a)$$

$$Ef(a) = e^{hD} f(a)$$

$$\Rightarrow E = e^{hD}$$

But  $E = (1 - \nabla)^{-1}$  ( $\because$  by backward difference operator.)

$$\Rightarrow (1 - \nabla)^{-1} = e^{hD}$$

$$\Rightarrow hD = (-1) \ln(1 - \nabla)$$

$$= \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots$$

$$D = \frac{1}{h} \left( \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots \right)$$

So in general;

$$D^n = \frac{1}{h^n} \left( \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots \right)^n$$

For  $n=1$

$$Df(a) = \frac{1}{h} \left( \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots \right) f(a)$$



For  $n=2$ 

$$\begin{aligned}
 D^2 f(a) &= \frac{1}{h^2} (\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \dots)^2 f(a) \\
 &= \frac{1}{h^2} (\nabla^2 + \frac{\nabla^4}{4} + \frac{\nabla^3}{2} + \frac{\nabla^4}{3} + \frac{\nabla^4}{3} + \\
 &\quad \frac{1}{4} \nabla^5 + \frac{1}{6} \nabla^5 + \frac{1}{6} \nabla^5 + \frac{1}{4} \nabla^5 + \dots) \\
 &= \frac{1}{h^2} (\nabla^2 + \nabla^3 + \frac{11}{12} \nabla^4 + \frac{5}{6} \nabla^5 + \dots) f(a)
 \end{aligned}$$

For  $n=3$ 

$$\begin{aligned}
 D^3 f(a) &= \frac{1}{h^3} (\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + \dots)^3 f(a) \\
 &= \frac{1}{h^3} (\nabla^3 + (\frac{1}{2} + 1) \nabla^4 + (\frac{1}{3} + \frac{1}{2} + \frac{11}{12}) \nabla^5 + \\
 &\quad (\frac{1}{4} + \frac{1}{3} + \frac{11}{24} + \frac{5}{6}) \nabla^6 + \dots) f(a) \\
 &= \frac{1}{h^3} [\nabla^3 + \frac{3}{2} \nabla^4 + \frac{7}{4} \nabla^5 + \frac{15}{8} \nabla^6 + \dots] f(a)
 \end{aligned}$$

Remark:-

If we wish to calculate the derivative of a function at a point which is given in beginning of table, we use forward difference formula and if we wish to calculate derivative near end points of the data, then we use some backward difference formula for differentiation.

Example:-

Find  $f'(4)$ ,  $f''(4)$  from table

|          |   |   |    |    |     |     |
|----------|---|---|----|----|-----|-----|
| $x$ :    | 0 | 1 | 2  | 3  | 4   | 5   |
| $f(x)$ : | 3 | 4 | 19 | 84 | 259 | 628 |

Solution:-

The difference table is:

| $x$ | $f(x)$ | $\nabla$ | $\nabla^2$ | $\nabla^3$ | $\nabla^4$ | $\nabla^5$ |
|-----|--------|----------|------------|------------|------------|------------|
| 0   | 3      |          |            |            |            |            |
|     |        | 1        |            |            |            |            |
| 1   | 4      |          | 14         |            |            |            |
|     |        | 15       |            | 36         |            |            |
| 2   | 19     |          | 50         |            | 24         |            |
|     |        | 65       |            | 60         |            | 0          |
| 3   | 84     |          | 110        |            | 24         |            |
|     |        | 175      |            | 84         |            |            |
| → 4 | 259    |          | 194        |            |            |            |
|     |        | 369      |            |            |            |            |
| 5   | 628    |          |            |            |            |            |

Now Here  $a=4$ 

The first derivative of the function is given by.

$$\begin{aligned}
 Df(4) &= \frac{1}{h} [\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots] f(4) \\
 &= \frac{1}{1} [\nabla f(4) + \frac{\nabla^2 f(4)}{2} + \frac{\nabla^3 f(4)}{3} + \frac{\nabla^4 f(4)}{4}] \\
 &= 1 [175 + \frac{1}{2}(110) + \frac{1}{3}(60) + \frac{1}{4}(24)]
 \end{aligned}$$

$$f'(4) = 175 + 55 + 20 + 6$$

$$= 256$$

2<sup>nd</sup> derivative of function at  $x=4$ .

$$f''(4) = \frac{1}{h^2} [\Delta^2 + \Delta^3 + \frac{11}{12} \Delta^4 + \frac{5}{6} \Delta^5 + \dots] f(4)$$

$$= \frac{1}{12} [110 + 60 + \frac{11}{12} (24)]$$

$$= 110 + 60 + 22$$

$$f''(4) = 192$$

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Question:-

Compute first three derivative of  $f(x)$   
at  $x=2$  from data

$$x: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$

$$y: -6 \quad -10 \quad -8 \quad 6 \quad 38 \quad 94$$

Solution:- The difference table is

| $x$ | $y$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|-----|-----|----------|------------|------------|------------|
| 0   | -6  |          |            |            |            |
|     |     | -4       |            |            |            |
| 1   | -10 |          | 6          |            |            |
|     |     | 2        |            | 6          |            |
| → 2 | -8  |          | 12         |            | 0          |
|     |     | 14       |            | 6          |            |
| 3   | 6   |          | 8          |            | 0          |
|     |     | 32       |            | 6          |            |
| 4   | 38  |          | 24         |            |            |
| 5   | 94  |          |            |            |            |

Here  $a=2$   
First derivative is given by

$$Df(a) = \frac{1}{h} (\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots) f(a)$$

$$= \frac{1}{1} (\Delta f(2) - \frac{\Delta^2 f(2)}{2} + \frac{\Delta^3 f(2)}{3} - \frac{\Delta^4 f(2)}{4} + \dots)$$

$$= 14 - \frac{1}{2} (18) + \frac{1}{3} (6)$$

$$= 14 - 9 + 2 = 7$$

2<sup>nd</sup> derivative is

$$D^2 f(a) = \frac{1}{h^2} [\Delta^2 - \frac{\Delta^3}{2} + \dots] f(a)$$

$$= \frac{1}{1^2} [\Delta^2 f(2) - \frac{\Delta^3 f(2)}{2} + \dots]$$

$$= 12 - \frac{1}{2} (18) = -4$$

3<sup>rd</sup> derivative is

$$D^3 f(a) = \frac{1}{h^3} [\Delta^3 - \frac{3}{2} \Delta^4 + \dots] f(a)$$

$$f'''(2) = \frac{1}{1} [\Delta^3 f(2)]$$

$$f'''(2) = 6$$

(Answer)

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Exercise

- 1) Find first three derivative of  $f(x)$  at  $x=2.2$
- |        |    |    |    |    |
|--------|----|----|----|----|
| $x$    | 0  | 2  | 3  | 4  |
| $f(x)$ | -2 | 16 | 40 | 82 |

Ans:  $f' = 19.52$   
 $f'' = 13.2$   
 $f''' = 6$

- 2) Using backward difference operator formula find three first derivative.
- |        |   |   |   |    |    |     |
|--------|---|---|---|----|----|-----|
| $x$    | 0 | 1 | 2 | 3  | 4  | 5   |
| $f(x)$ | 2 | 2 | 8 | 26 | 62 | 122 |

Ans:  $f' = 9$ ,  $f'' = 6$   
 $f''' = 0$

- 3) Find minimum value of  $f(x)$  from table

|        |        |        |        |        |
|--------|--------|--------|--------|--------|
| $x$    | 0.60   | 0.65   | 0.70   | 0.75   |
| $f(x)$ | 0.6221 | 0.6155 | 0.6138 | 0.6170 |

Ans:  $x = 0.692$   
 $y_{\min} = 0.6137$

- 4) Calculate  $f'(x)$  and  $f''(x)$  at  $x=1.2$   
 the value of  $f(x) = \tan^{-1}(e^x)$   
 tabulated below

|                  |         |         |         |         |         |
|------------------|---------|---------|---------|---------|---------|
| $x$              | 1       | 1.1     | 1.2     | 1.3     | 1.4     |
| $\tan^{-1}(e^x)$ | 1.21828 | 1.24946 | 1.27824 | 1.30473 | 1.32982 |

- 5) Find first two derivative at  $x=4$
- |        |   |    |    |    |     |     |     |
|--------|---|----|----|----|-----|-----|-----|
| $x$    | 1 | 2  | 3  | 4  | 5   | 6   | 7   |
| $f(x)$ | 4 | 11 | 20 | 67 | 128 | 219 | 346 |

Ans: 48, 24

- 6) Find radius of curvature at 0.6 on curve defined by following

|     |      |       |       |       |       |       |       |
|-----|------|-------|-------|-------|-------|-------|-------|
| $x$ | 0.0  | 0.2   | 0.4   | 0.6   | 0.8   | 1.0   | 1.2   |
| $y$ | 0.70 | 1.175 | 1.811 | 2.666 | 3.801 | 5.292 | 7.232 |

Hint: Find  $y'(0.6)$  and  $y''(0.6)$  and put in

$$P = \frac{\{1 + (y')^2\}^{3/2}}{y''}$$

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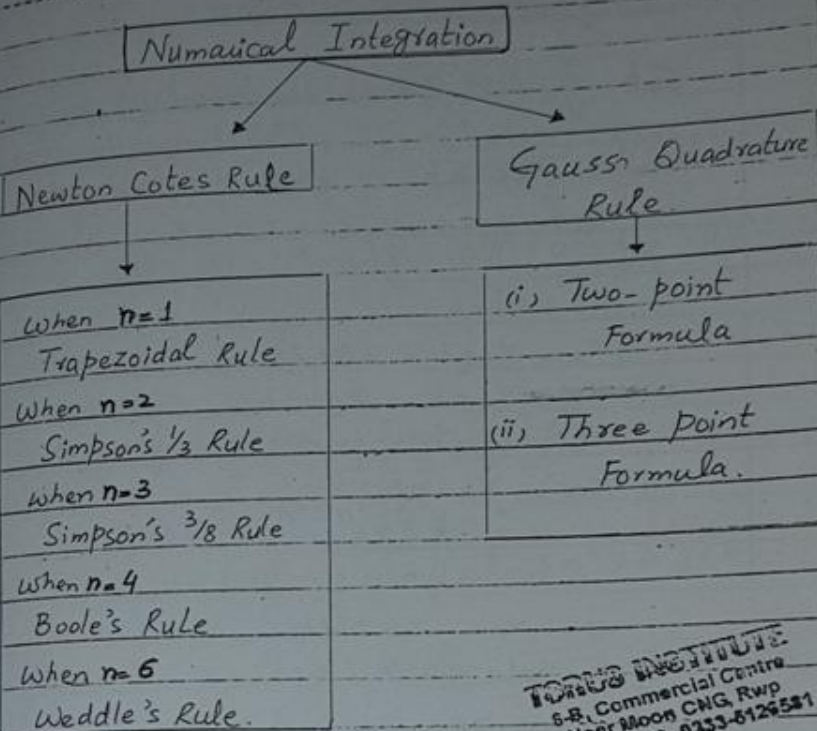
# NUMERICAL INTEGRATION

Sometime it is either difficult or impossible to evaluate the definite integral by using well known analytic method e.g.

$$\int_0^{\pi} \frac{x + \tan^{-1}(x)}{\ln(1 + \sin x) + e^{-x^2}} dx$$

In such cases we use Numerical integration which gives us an approximation to the value of defined integration. There are several method available for numerical integration, but the most commonly used methods be classified in two groups:

- (i) Newton - Cote's Rule
- (ii) Gauss - Quadrature Rule.



Where n is number of intervals.

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**TOORU INSTITUTE**  
 6-B, Commercial Centre  
 Near Moon CNG Rwp  
 Ph: 4421548, 0253-6126521

(1) NEWTON-COTES RULE:

Let  $y = f(x)$  takes the values  $y_0, y_1, \dots, y_n$  or  $(f(x_0), f(x_1), \dots, f(x_n))$  equally spaced arguments at  $x_0, x_1, x_2, \dots, x_n$  resp.

Let  $x_1 - x_0 = x_2 - x_1 = h$

Then

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h = x_0 + 2h$$

$$x_3 = x_2 + h = x_0 + 3h$$

$$\dots \dots \dots$$

$$x_n = x_0 + nh$$

where  $u = \frac{x - x_0}{h}$

$$\Rightarrow x = x_0 + uh$$

$$dx = h du$$

When  $x = x_0$  ,  $x = x_n$

$$u = 0$$

$$u = \frac{x_n - x_0}{h} = \frac{nh}{h} = n$$

$$\text{Now } \int_{x_0}^{x_n} f(x) dx = \int_0^n f(x_0 + uh) h du$$

$$= h \int_0^n f(x_0 + uh) du$$

$$= h \int_0^n (1 + \Delta)^u f(x_0) du \quad \begin{matrix} E^n f(x_0) \\ \therefore (1+\Delta)^u f(x_0) \end{matrix}$$

$$= h \int_0^n \left[ f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(x_0) + \dots \right] du$$

$\therefore$  by Newton's forward difference

$$= h \int_0^n \left[ y_0 + u \Delta y_0 + \frac{u^2 - u}{2} \Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{6} \Delta^3 y_0 + \frac{u^4 - 6u^3 + 11u^2 - 6u}{24} \Delta^4 y_0 + \dots \right] du$$

$$= h \left[ y_0 u \Big|_0^n + \Delta y_0 \frac{u^2}{2} \Big|_0^n + \frac{\Delta^2 y_0}{2} \left( \frac{u^3}{3} - \frac{u^2}{2} \right) \Big|_0^n + \frac{\Delta^3 y_0}{6} \left( \frac{u^4}{4} - u^3 + u \right) \Big|_0^n + \frac{\Delta^4 y_0}{24} \left( \frac{u^5}{5} - \frac{6u^4}{4} + \frac{11u^3}{3} - 6u^2 \right) \Big|_0^n + \dots \right]$$

$$= h \left[ n y_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2} \left[ \frac{n^3}{3} - \frac{n^2}{2} \right] \Delta^2 y_0 + \frac{1}{6} \left( \frac{n^4}{4} - n^3 + n \right) \Delta^3 y_0 + \frac{1}{24} \left( \frac{n^5}{5} - \frac{3}{2} n^4 + \frac{11}{3} n^3 - 3n^2 \right) \Delta^4 y_0 + \dots \right]$$

This is called Newton Cote's formula.

(i) Trapezoidal Rule:-

In order to evaluate  $\int_a^b f(x) dx$ ; Let us divide  $[a, b]$  into 'n' subintervals each of length 'h' where  $h = \frac{b-a}{n}$ , then

$P = \{x_0, x_1, x_2, \dots, x_n\}$  is partition of  $[a, b]$

with  $x_k - x_{k-1} = h \quad \forall k$

Now for all



$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx \rightarrow (*)$$

1<sup>st</sup> we Consider  $\int_{x_0}^{x_1} f(x) dx$ .

In this case  $f(x)$  is a polynomial of degree 1. So this has first forward difference is constant and all higher order differences are zero.

Now put  $n=1$  in Newton Cote's formula

$$\int_{x_0}^{x_1} f(x) dx = h \left[ y_0 + \frac{1}{2} \Delta y_0 + 0 + 0 + \dots + 0 \right]$$

$$\begin{aligned} &= h \left[ y_0 + \frac{1}{2} (y_1 - y_0) \right] = \Delta y_0 = y_1 - y_0 \\ &= \frac{h}{2} [2y_0 + y_1 - y_0] \\ &= \frac{h}{2} (y_0 + y_1) \end{aligned}$$

Similarly;  $\int_{x_1}^{x_2} f(x) dx = \frac{h}{2} (y_1 + y_2)$

$$\int_{x_2}^{x_3} f(x) dx = \frac{h}{2} (y_2 + y_3)$$

$$\dots \dots \dots$$

$$\int_{x_{n-1}}^{x_n} f(x) dx = \frac{h}{2} (y_{n-1} + y_n)$$

Put in (\*) we get

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$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n)$$

which is trapezoidal Rule.

(ii) Simpson's ( $\frac{1}{3}$ ) Rule:-

In order to evaluate  $\int_{x_0}^{x_n} f(x) dx$ , let divide  $[a, b]$  into 'n' subintervals where n should be even

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-2}}^{x_n} f(x) dx \rightarrow (*)$$

1<sup>st</sup> we find  $\int_{x_0}^{x_2} f(x) dx$  ( $\because n=2$ )

In this case  $f(x)$  is polynomial of degree 2. so that's first and 2<sup>nd</sup> forward difference are constant and higher order difference are zero.

Put  $n=2$  in Newton Cote's formula

$$\begin{aligned} \int_{x_0}^{x_2} f(x) dx &= h \left[ 2y_0 + \frac{4}{2} \Delta y_0 + \frac{1}{2} \left( \frac{8-4}{3} \right) \Delta^2 y_0 + 0 + 0 \dots \right] \\ &= h \left[ 2y_0 + 2\Delta y_0 + \frac{1}{3} \Delta^2 y_0 \right] \end{aligned}$$

$$= h \left[ 2(y_1 - y_0) + \frac{1}{3} (y_2 - 2y_1 + y_0) \right]$$

$$= h \left[ 2y_0 + 2y_1 - 2y_0 + \frac{1}{3} y_2 - \frac{2}{3} y_1 + \frac{1}{3} y_0 \right]$$



$$= \frac{h}{3} (6y_1 + y_0 - 2y_1 - y_0)$$

$$= \frac{h}{3} (y_0 + 4y_1 + y_2)$$

Similarly;  $\int_{x_2}^{x_4} f(x) dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$

$$\int_{x_4}^{x_6} f(x) dx = \frac{h}{3} (y_4 + 4y_5 + y_6)$$

$$\int_{x_{n-2}}^{x_n} f(x) dx = \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n)$$

So by (+)

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n]$$

This formula is called Simpson's  $1/3$  Rule.

### (iii) Simpson's $(3/8)$ Rule:-

In order to evaluate  $\int_a^b f(x) dx$ . Let us denote

( $n$  is multiple of 3) and divide  $[a, b]$  into intervals.

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_3} f(x) dx + \int_{x_3}^{x_6} f(x) dx + \dots + \int_{x_{n-3}}^{x_n} f(x) dx \rightarrow (*)$$

Consider  $\int_{x_0}^{x_3} f(x) dx$  to find that integral.

Put  $n=3$  in Newton Cote's formula we have

$$\int_{x_0}^{x_3} f(x) dx = 3h \left[ y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right]$$

$$= \frac{3h}{8} [8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0]$$

$$= \frac{3h}{8} [8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8} [8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + 3y_1 + y_3 - 3y_2 - y_0]$$

$$= \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$$

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly;

$$\int_{x_3}^{x_6} f(x) dx = \frac{3h}{8} [y_3 + 3y_4 + 3y_5 + y_6]$$

$$\int_{x_6}^{x_9} f(x) dx = \frac{3h}{8} [y_6 + 3y_7 + 3y_8 + y_9]$$

$$\int_{x_{n-3}}^{x_n} f(x) dx = \frac{3h}{8} [y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Put in (\*), we have

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} \left[ y_0 + 3(y_1 + y_2 + y_4 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots + y_{n-3}) + y_n \right]$$

Which is Simpson's ( $\frac{3}{8}$ ) Rule.

#### (iv) Boole's Rule :-

To evaluate  $\int_a^b f(x) dx$   
we partitioned interval  $[a, b]$   
where  $n = 4$  (or 5 points)

$f(x)$  is a polynomial of degree 4  
and all differences of order higher  
than 4 are zero.

Put  $n = 4$  in Newton Cote's formula.

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_4} f(x) dx + \int_{x_4}^{x_8} f(x) dx + \dots + \int_{x_{n-4}}^{x_n} f(x) dx$$

$x_{n-4} \rightarrow (*)$

First find  $\int_{x_0}^{x_4} f(x) dx$ . (then put  $n = 4$   
in N.C.F)

$$\begin{aligned} \int_{x_0}^{x_4} f(x) dx &= 4h \left[ y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0 \right] \\ &= 4h \left[ y_0 + 2(y_1 - y_0) + \frac{5}{3}(y_2 - 2y_1 + y_0) + \frac{2}{3}(y_3 - 3y_2 + 3y_1 - y_0) + \frac{7}{90}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) \right] \end{aligned}$$

$$= \frac{2h}{45} \left[ 90y_0 + 18(y_1 - y_0) + 150(y_2 - 2y_1 + y_0) + 60(y_3 - 3y_2 + 3y_1 - y_0) + 7(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) \right]$$

$$= \frac{2h}{45} [7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

Similarly;

$$\int_{x_4}^{x_8} f(x) dx = \frac{2h}{45} [7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$\int_{x_{n-4}}^{x_n} f(x) dx = \frac{2h}{45} [7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n]$$

Put all in (\*) we get

$$\begin{aligned} \int_{x_0}^{x_n} f(x) dx &= \frac{2h}{45} \left[ 7(y_0 + y_n) + 32(y_1 + y_3 + y_5 + \dots + y_{n-1}) \right. \\ &\quad \left. + 12(y_2 + y_6 + y_{10} + \dots + y_{n-2}) \right. \\ &\quad \left. + 14(y_4 + y_8 + y_{12} + \dots + y_{n-4}) \right] \end{aligned}$$

Which is required Boole's Rule.



### Weddle's Rule:-

In order to evaluate  $\int_a^b f(x) dx$ . Let divide  $[a, b]$  into subinterval  $P = \{x_0, x_1, x_2, \dots, x_n\}$

For all  $x_n$

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx$$

First find  $\int_{x_0}^{x_n} f(x) dx$ .

In this case  $f(x)$  is polynomial of degree 6 and all forward difference of greater than 6 are zero.

Put  $n=6$  in Newton Cote's formula

$$\int_{x_0}^{x_6} f(x) dx = h \left[ 6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0 \right]$$

$$= \frac{3h}{10} \left[ 20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \Delta^6 y_0 \right]$$

$$= \frac{3h}{10} \left[ 20y_0 + 6(y_1 - y_0) + 90(y_2 - 2y_1 + y_0) + 8(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0) + (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0) \right]$$

$$= \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly:

$$\int_{x_6}^{x_{12}} f(x) dx = \frac{3h}{10} [y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$\int_{x_{n-6}}^{x_n} f(x) dx = \frac{3h}{10} [y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Put in (\*) we have

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{10} \left[ (y_0 + y_2 + y_4 + y_6 + \dots + y_n) + 2(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 5(y_2 + y_4 + y_6 + \dots + y_{n-2}) + 6(y_3 + y_5 + y_7 + \dots + y_{n-3}) \right]$$

Which is required weddle's rule.

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Question:- Evaluate  $\int_0^1 \frac{1}{1+x^2}$  using  
 (i) Trapezoidal rule with 5 points  
 (ii) Simpson's Rule 5 points  
 (iii) Boole's Rule.

Compare result with exact value.

Solution:-

$$a = 0, b = 1, n = 4$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

$$f(x) = \frac{1}{1+x^2}$$

|        | $x_0 =$ | $x_1 =$ | $x_2 =$ | $x_3 =$ | $x_4 =$ |
|--------|---------|---------|---------|---------|---------|
| $x$    | 0       | 0.25    | 0.5     | 0.75    | 1       |
| $f(x)$ | 1       | 0.941   | 0.8     | 0.64    | 0.5     |

(i) Trapezoidal Rule:-

$$\int_{x_0}^{x_4} f(x) dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3) + y_4)$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{2} [1 + 2(0.941 + 0.8 + 0.64) + 0.5]$$

$$= \frac{0.25}{2} (6.282) = 0.78275$$

(ii) Simpson's  $\frac{1}{3}$  Rule:-

$$\int_{x_0}^{x_4} f(x) dx = \frac{h}{3} [y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{3} [1 + 4(0.941 + 0.64) + 2(0.8) + 0.5]$$

$$= 0.78533$$

(iii) Boole's Rule:-

$$\int_{x_0}^{x_4} f(x) dx = \frac{2h}{45} [7(y_0 + y_1) + 32(y_1 + y_3) + 12(y_2)]$$

$$= \frac{2(0.25)}{45} [7(1 + 0.5) + 32(0.941 + 0.64) + 12(0.8)]$$

$$= 0.785466$$

Exact Value:

$$\int_0^1 \frac{1}{1+x^2} dx = \tan^{-1}(x) \Big|_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

$$= 0.78539$$

Error in Simpson's Rule is smallest  
 So Simpson's Rule is better rule  
 other than rules.

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(Answer)  
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 6-B, Commercial Centre  
 Near Moon Chng, Rwp  
 Ph: 4421549, 0333-51265

Example:-

Apply Simpson's  $\frac{1}{3}$  rule to find approx. value of  $\pi$  from  $\int_0^1 \frac{1}{1+x^2} dx$  taking 4 intervals.

Solution:- 
$$I = \int_0^1 \frac{1}{1+x^2} dx$$

$$= \tan^{-1}(x) \Big|_0^1$$

$$= \frac{\pi}{4} - 0$$

$$\Rightarrow I = \pi/4$$

$$\Rightarrow \pi = 4I$$

Now we find approximate value of  $I$  using Simpson's  $\frac{1}{3}$  Rule.

Take  $a = 0$

$$b = 1$$

$$n = 4$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

So table is

|        |   |        |      |       |      |
|--------|---|--------|------|-------|------|
| $x$    | 0 | 0.25   | 0.50 | 0.75  | 1.0  |
| $f(x)$ | 1 | 0.9412 | 0.80 | 0.640 | 0.50 |

Now by Simpson's  $\frac{1}{3}$  Rule

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{0.25}{3} [1 + 5 + 4(0.9412 + 0.6400) + 2(0.80)]$$

$$I = 0.7854$$

So approximate value of  $\pi$  is

$$\pi = 4I$$

$$= 4(0.7854)$$

$$= 3.1416$$

(Answer).

Example: Evaluate  $\int_0^{\pi/2} \sin x dx$  using boole's Rule.

Solution:-

Here  $a = 0$ ,  $b = \pi/2$

$$f(x) = \sin x$$

Take  $n = 4$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

So table is

| $x$      | $f(x) = y$ |
|----------|------------|
| 0        | 0          |
| $\pi/8$  | 0.3827     |
| $\pi/4$  | 0.7071     |
| $3\pi/8$ | 0.9239     |
| $\pi/2$  | 1          |

By Boole's Rule :-

$$\int_0^{\pi/2} \sin x dx = \frac{2h}{45} [7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2(\pi/8)}{45} [7(0) + 32(0.3827) + 12(0.7071) + 32(0.9239) + 7(1)]$$

$$= \frac{\pi}{180} (57.2964)$$

$$\int_0^{\pi/2} \sin x dx = 1.000$$

(Answer)

Example:

Use Boole's formula and  $3/8$  Simpson Rule in combination to evaluate integral  $\int_0^{0.7} f(x) dx$  when  $f(x)$  given by

|       |   |        |        |       |        |        |
|-------|---|--------|--------|-------|--------|--------|
| $x :$ | 0 | 0.1    | 0.2    | 0.3   | 0.4    | 0.5    |
| $y :$ | 0 | 0.001  | 0.399  | 0.893 | 0.1579 | 0.2449 |
|       |   | 0.6    | 0.7    |       |        |        |
|       |   | 0.3497 | 0.4705 |       |        |        |

Solution :

$$a = 0$$

$$b = 0.7$$

$$h = 0.1 \text{ (as from table)}$$

The given function is

|       | $x$ | $y$    |
|-------|-----|--------|
| $x_0$ | 0   | 0      |
| $x_1$ | 0.1 | 0.001  |
| $x_2$ | 0.2 | 0.399  |
| $x_3$ | 0.3 | 0.893  |
| $x_4$ | 0.4 | 0.1579 |
| $x_5$ | 0.5 | 0.2449 |
| $x_6$ | 0.6 | 0.3497 |
| $x_7$ | 0.7 | 0.4705 |

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In Simpson's  $\frac{3}{8}$  Rule  $n=3$  and in boole's Rule  $n=4$ .

So there are two possibilities.

(i) From  $x_0$  to  $x_3$  apply Simpson's  $3/8$  and  $x_3$  to  $x_7$  apply Boole's rule then add it.

(ii) From  $x_0$  to  $x_4$  apply Boole's Rule and  $x_4$  to  $x_7$  apply Simpson's  $3/8$  Rule then add it.

$$\int_0^{0.7} f(x) dx = \frac{2h}{45} [7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$+ \frac{3h}{8} [y_4 + 3(y_5 + y_6) + y_7]$$

$$= \frac{2(0.1)}{45} [7(0) + 32(0.001) + 12(0.399) + 32(0.893) + 7(0.1579)]$$

$$+ \frac{3(0.1)}{8} [0.1579 + 3(0.2449 + 0.3497) + 0.4705]$$



$$\int_0^{0.7} f(x) dx = \frac{0.2}{45} (4.4737) + \frac{0.3}{8} (2.4122)$$

$$= 0.0199 + 0.0905$$

$$= 0.1104$$

(Answer)

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## Exercise

1): Evaluate  $\int_0^1 (1+x+e^x) dx$  using trapezoidal rule for 8 - trapeziums. upto 5 decimal - places.

2): Evaluate  $\int_0^{\pi} \frac{\sin x}{x} dx$  for  $n=6$  by Simpson's  $^o$  rule.

3): Integrate function  $f(x)$  given by following table i, using 5 pts Simpson's Rule  
 (ii) Boole's Rule

|        |   |      |      |      |      |
|--------|---|------|------|------|------|
| $x$    | 1 | 1.5  | 2.0  | 2.5  | 3.0  |
| $f(x)$ | 1 | 0.44 | 0.25 | 0.16 | 0.11 |

4): Evaluate  $\int_0^1 \frac{1}{1+x^4} dx$  using  
 (i) Trapezoidal Rule  
 (ii) Simpson's  $^{3/8}$  Rule

Compare result with exact value and which of two method is superior and why.

5): Use 7 points weddle rule to evaluate  $\int_0^{\pi/2} \sin x dx$  using 6 digits accuracy.

## (II) GAUSS QUADRATURE FORMULA

Gauss investigated that in the quadrature formulae, the equally spaced arguments make the accuracy of quadrature formula limited.

He derived a quadrature formula in which the limit of integration are from -1 to +1, if not then these can be made by putting

$$x = \frac{(b-a)t + (a+b)}{2}$$

$$\Rightarrow t = \frac{2x - a - b}{b - a}$$

$$\text{at } x = a, t = \frac{2a - a - b}{b - a} = \frac{a - b}{b - a} = -1$$

$$\text{at } x = b, t = \frac{2b - a - b}{b - a} = \frac{b - a}{b - a} = +1$$

### § Derivation of Gauss Two point Quadrature Formula :-

$$\text{Let } I = \int_a^b f(x) dx$$

$$\text{Put } x = \frac{(b-a)t + (b+a)}{2}$$

$$dx = \frac{b-a}{2} dt$$

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$$\text{When } x = a, t = -1$$

$$x = b, t = 1$$

$$\text{In this case } I = \int_a^b f(x) dx = \int_{-1}^1 f\left(\frac{(b-a)t + (b+a)}{2}\right) \frac{(b-a)}{2} dt$$

$$= \frac{b-a}{2} \int_{-1}^1 F(t) dt \rightarrow \text{vi}$$

$$\text{where } F(t) = f\left(\frac{(b-a)t + (b+a)}{2}\right)$$

$$\text{Let } \int_{-1}^1 F(t) dt = C_1 F(t_1) + C_2 F(t_2) \rightarrow \text{vii}$$

Two points are  $t_1$  and  $t_2$ .

Where  $C_1, C_2, t_1$  and  $t_2$  are unknown and  $C_1 \neq 0, C_2 \neq 0$  and  $t_1 \neq t_2$ .

Now as, cubic polynomial can be constructed at the most by using (4) unknown quantities so, to determine 4 unknown quantities.

$$\text{We take } F(t) = 1, t, t^2, t^3$$

When  $F(t) = 1$  : then

$$\int_{-1}^1 F(t) dt = C_1 F(t_1) + C_2 F(t_2)$$

$$\Rightarrow \int_{-1}^1 1 dt = C_1 (1) + C_2 (1)$$

$$2 = C_1 + C_2 \rightarrow \text{(iii)}$$

When  $F(t) = t$  : then (ii) gives

$$\int_{-1}^1 t dt = C_1 t_1 + C_2 t_2$$

$$\Rightarrow 0 = C_1 t_1 + C_2 t_2 \rightarrow \text{(iv)}$$

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When  $F(t) = t^2$ : then (iii) gives

$$\int_{-1}^1 t^2 dt = C_1 t_1^2 + C_2 t_2^2$$

$$\left[ \frac{t^3}{3} \right]_{-1}^1 = C_1 t_1^2 + C_2 t_2^2$$

$$\frac{2}{3} = C_1 t_1^2 + C_2 t_2^2 \rightarrow (v)$$

When  $F(t) = t^3$ : then (iii) gives

$$\int_{-1}^1 t^3 dt = C_1 t_1^3 + C_2 t_2^3$$

$$\left[ \frac{t^4}{4} \right]_{-1}^1 = C_1 t_1^3 + C_2 t_2^3$$

$$\Rightarrow 0 = C_1 t_1^3 + C_2 t_2^3 \rightarrow (vi)$$

Now multiplying (4) by  $t_1^2$  and then subtracting from eq. (5)

$$C_1 t_1^3 + C_2 t_2^3 = 0$$

$$\pm C_1 t_1^3 \pm C_2 t_2 t_1^2 = 0$$

$$C_2 t_2 (t_2^2 - t_1^2) = 0$$

$$C_2 t_2 (t_2 - t_1)(t_2 + t_1) = 0$$

$$\Rightarrow C_2 = 0 \text{ or } t_2 = 0 \text{ or } t_2 = t_1 \text{ or } t_2 = -t_1$$

but  $C_2 \neq 0$  and  $t_2 \neq t_1$

So  $t_2 = 0$  or  $t_2 = -t_1$

When  $t_2 = 0$  from (iv)  $C_1 t_1 = 0$

$$\Rightarrow C_1 = 0 \text{ or } t_1 = 0$$

but  $C_1 \neq 0$

So  $t_1 = 0$

But  $t_1 = t_2$  which is not possible.

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Hence  $t_2 \neq 0$ .

Hence finally  $t_2 = -t_1$

From (v)  $C_1 t_1^2 + C_2 (-t_1)^2 = \frac{2}{3}$

$$\Rightarrow (C_1 + C_2) t_1^2 = \frac{2}{3}$$

$$\Rightarrow 2 t_1^2 = \frac{2}{3} \quad (\because \text{by (iii)})$$

$$t = \pm \frac{1}{\sqrt{3}}$$

When  $t_1 = \frac{-1}{\sqrt{3}}$  then  $t_2 = \frac{1}{\sqrt{3}}$

When  $t_2 = \frac{-1}{\sqrt{3}}$  then  $t_1 = \frac{1}{\sqrt{3}}$

Next Put  $t_2 = -t_1$  in (iv)

$$C_1 t_1 - C_2 t_1 = 0$$

$$(C_1 - C_2) t_1 = 0$$

$$\Rightarrow C_1 - C_2 = 0 \quad \because t_1 \neq 0$$

$$\Rightarrow C_1 = C_2$$

From (iii)  $C_1 + C_2 = 2$

$$2 C_1 = 2$$

$$C_1 = 1$$

$$\Rightarrow C_2 = 1$$

So from (ii)

$$\int_{-1}^1 F(t) dt = (1) F\left(\frac{1}{\sqrt{3}}\right) + F\left(-\frac{1}{\sqrt{3}}\right)$$

$$\text{Hence } I = \int_a^b F(x) dx = \frac{b-a}{2} \left[ F\left(\frac{1}{\sqrt{3}}\right) + F\left(-\frac{1}{\sqrt{3}}\right) \right]$$

which is required Gauss two point Quadrature formula.

(proved)

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\* Question:-

Evaluate  $\int_0^{\pi/2} \sin x \, dx$  by two-point Gaussian formula.

Solution:-  $I = \int_0^{\pi/2} \sin x \, dx$

$$a = 0, \quad b = \pi/2$$

Put  $x = \frac{(b-a)t + (b+a)}{2}$

$$x = \frac{(\pi/2 - 0)t + (\pi/2 + 0)}{2}$$

$$x = \frac{\pi/2 t + \pi/2}{2}$$

$$dx = \frac{\pi}{4} dt$$

When  $x = 0$  ;  $t = -1$

When  $x = \frac{\pi}{2}$  ;  $t = 1$

$$I = \int_{-1}^1 \sin\left(\frac{\pi/2 t + \pi/2}{2}\right) \cdot \frac{\pi}{4} dt$$

$$= \frac{\pi}{4} \int_{-1}^1 F(t) dt \quad \text{where } F(t) = \sin\left(\frac{\pi t + \pi}{2}\right)$$

$$I = \frac{\pi}{4} \left[ F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \frac{\pi}{4} \left[ \sin\left(\frac{\pi/2(-1/\sqrt{3}) + \pi/2}{2}\right) + \sin\left(\frac{\pi/2(1/\sqrt{3}) + \pi/2}{2}\right) \right]$$

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$$= \frac{\pi}{4} [\sin(0.3319) + \sin(1.2388)]$$

$$= \frac{\pi}{4} [0.3258 + 0.94541]$$

$$= 0.99848$$

\* Question:-

Evaluate  $\int_0^1 e^{-x} dx$  using two-point Gauss-quad formula.

Solution:-

$$I = \int_0^1 e^{-x} dx$$

where  $a = 0$  ,  $b = 1$

Put  $x = \frac{(b-a)t + (b+a)}{2}$

$$x = \frac{1t + 1}{2}$$

$$x = \frac{t + 1}{2}$$

$$dx = \frac{1}{2} dt$$

When  $x = 0$  ;  $t = -1$

When  $x = 1$  ;  $t = 1$

$$I = \int_{-1}^1 e^{-\left(\frac{t+1}{2}\right)} \cdot \frac{1}{2} dt$$

$$= \frac{1}{2} \int_{-1}^1 e^{-\left(\frac{t+1}{2}\right)} dt$$

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by Gauss-quadrature formula

$$I = \frac{1}{2} \int_{-1}^1 F(t) dt \quad \text{where} \quad F(t) = e^{-\left(\frac{t+1}{2}\right)}$$

$$= \frac{1}{2} \left[ F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \frac{1}{2} \left[ e^{-\left(\frac{-1/\sqrt{3}+1}{2}\right)} + e^{-\left(\frac{1/\sqrt{3}+1}{2}\right)} \right]$$

$$= \frac{1}{2} \left[ e^{-0.2113} + e^{-0.7887} \right]$$

$$= \frac{1}{2} [0.8095 + 0.4544]$$

$$I = 0.6320 \quad (\text{Answer}).$$

\* GAUSS 3-point Quadrature Formula:  
(without prove)

$$\int_{-1}^1 F(x) dx = \frac{1}{9} \left[ 5F\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right) \right]$$

(Not include its  
proof.)

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Question:-

Apply Gauss Quadrature for 3-pts  
evaluate  $\int_{-1}^1 \frac{1}{1+x^2} dx$

Solution:-

$$I = \int_{-1}^1 \frac{1}{1+x^2} dx$$

$$= \int_{-1}^1 F(x) dx \quad ; \quad \text{where} \quad F(x) = \frac{1}{1+x^2}$$

$$I = \frac{1}{9} \left[ 5F\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right) \right]$$

$$= \frac{1}{9} \left[ 5 \cdot \frac{1}{1+3/5} + 8(1) + 5 \cdot \frac{1}{1+3/5} \right]$$

$$= \frac{1}{9} [5(0.625) + 8(1) + 5(0.625)]$$

$$= \frac{1}{9} (14.25)$$

$$I = 1.5833 \quad (\text{Ans})$$

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\* Question:

Apply 3-pt Gaussian Formula for

$$I = \int_2^3 \frac{1}{2 - \sin x^2} dx.$$

Solution:-  $I = \int_2^3 \frac{1}{2 - \sin x^2} dx$

where  $a = 2$ ,  $b = 3$ 

put  $x = \frac{(b-a)t + (a+b)}{2}$

$$x = \frac{(3-2)t + 5}{2}$$

$$x = \frac{1t + 5}{2}$$

$$dx = \frac{1}{2} dt$$

when  $x = 2$ ;  $t = -1$

when  $x = 3$ ;  $t = 1$

$$I = \int_{-1}^1 \frac{1}{2 - \sin\left(\frac{t+5}{2}\right)^2} \cdot \frac{1}{2} dt$$

$$= \frac{1}{2} \int_{-1}^1 \frac{1}{2 - \sin\left(\frac{t+5}{2}\right)^2} dt \rightarrow (*)$$

$$\int_{-1}^1 \frac{1}{2 - \sin\left(\frac{t+5}{2}\right)^2} dt = \frac{1}{9} \left[ F\left(-\sqrt{\frac{3}{5}}\right) \cdot 5 + 8F(0) + 5F\left(\sqrt{\frac{3}{5}}\right) \right]$$

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$$= \frac{1}{9} \left[ \frac{1 \cdot 5}{2 - \sin\left(\frac{\sqrt{3/5} + 5}{2}\right)^2} + 8\left(\frac{1}{2}\right) + \frac{5}{2 - \sin\left(\frac{\sqrt{3/5} + 5}{2}\right)^2} \right]$$

$$= \frac{1}{9} \left[ \frac{5}{2 - 0.7340} + \frac{8}{2} + \frac{5}{2 - 0.0633} \right]$$

$$= \frac{1}{9} (3.9494 + 4 + 2.5817)$$

$$= \frac{10.5311}{9}$$

$$= 1.1701$$

put in \*

$$I = \frac{1}{2} \int_{-1}^1 \frac{1}{2 - \sin\left(\frac{t+5}{2}\right)^2} dt$$

$$= \frac{1}{2} (1.1701) = 0.5851$$

(Answer)

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Exercise :

Apply 2 and 3 pts Gaussian Formulae.

(i)  $I = \int_0^1 \frac{dx}{1+x^2}$  Ans: 0.7868

(ii)  $I = \int_0^1 \frac{\sin x}{x} dx$  Ans: 0.9443

(iii)  $I = \int_0^1 x dx$  Ans: 0.5

(iv)  $I = \int_{-1}^1 (1-x^2)^{2/3} \cos x dx$  Ans: 1.32



# ERROR ANALYSIS

In Numerical Integration; we study many methods to evaluate problem and we demonstrated through examples that some methods are superior to others but we did not give any theoretical justification.

Now we shall calculate the amount of error in each method, for this we shall use Taylor's Series.

Note:

Error =  $X - A$  where

$X$  = Exact value

$A$  = Approximate value

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## (1) Error Analysis of Trapezoidal Rule ; $n=1$

$$\begin{aligned}
 E &= X - A \\
 &= \int_{x_0}^{x_1} f(x) dx - \frac{h}{2} (f_0 + f_1) \\
 &= \int_{x_0}^{x_0+h} f(x) dx - \frac{h}{2} [f_0 + f(x_0+h)] \\
 &= F(x_1) - F(x_0) - \frac{h}{2} [f_0 + f(x_0+h)] \\
 &= F(x_0+h) - F(x_0) - \frac{h}{2} [f_0 + f(x_0+h)] \\
 &= \cancel{F(x_0)} + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \dots - \cancel{F(x_0)} \\
 &\quad - \frac{h}{2} [f_0 + (f_0 + hf_0' + \frac{h^2}{2!}f_0'' + \dots)] \\
 &= \cancel{hf_0} + \frac{h^2}{2!}f_0' + \frac{h^3}{3}f_0'' + \dots - \cancel{\frac{h}{2}f_0} - \cancel{\frac{h}{2}f_0} \\
 &\quad - \frac{h^2}{2}f_0' - \frac{h^3}{2.2!}f_0'' - \frac{h^4}{2.3!}f_0''' - \dots \\
 &= \left( \frac{1}{3!} - \frac{1}{2.2!} \right) h^3 f_0'' + \dots \\
 &= \left( \frac{1}{6} - \frac{1}{4} \right) h^3 f_0'' \\
 E &= -\frac{1}{12} h^3 f_0'' \quad \text{or} \quad -\frac{1}{12} h^3 f''(x_0)
 \end{aligned}$$

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## (II) Error Analysis of Simpson's $\frac{1}{3}$ Rule

$n=2$ .

$$E = X - A$$

$$= \int_{x_0}^{x_2} f(x) dx - \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2))$$

$$E = I_1 - I_2 \rightarrow (i)$$

$$I_1 = \int_{x_0}^{x_2} f(x) dx \quad \text{Let } f(x) = F'(x)$$

$$= \int_{x_0}^{x_2} F'(x) dx$$

$$= [F(x)]_{x_0}^{x_2}$$

$$= F(x_2) - F(x_0)$$

$$= F(x_0 + 2h) - F(x_0)$$

$$= F(x_0) + 2h F'(x_0) + \frac{4h^2}{2!} F''(x_0) + \frac{9h^3}{3!} F'''(x_0)$$

$$+ \dots - F(x_0)$$

$$I_1 = 2h f(x_0) + 2h^2 f'(x_0) + \frac{4}{3} h^3 f''(x_0) + \frac{2}{3} h^4 f'''(x_0) + \frac{4}{15} h^5 f^{(4)}(x_0) + \dots$$

$$I_2 = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$

$$= \frac{h}{3} [f(x_0) + 4f(x_0 + h) + f(x_0 + 2h)]$$

$$= \frac{h}{3} \left[ f(x_0) + 4 \left\{ f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \dots \right\} + f(x_0) + 2h f'(x_0) + \frac{4h^2}{2!} f''(x_0) + \frac{9h^3}{3!} f'''(x_0) + \dots \right]$$

$$= 2h f(x_0) + 2h^2 f'(x_0) + \frac{9}{3} h^3 f''(x_0) + \frac{2}{3} h^4 f'''(x_0) + \frac{4}{15} h^5 f^{(4)}(x_0) + \dots$$

Put  $I_1$  and  $I_2$ 's value in (i)

$$E = -\frac{1}{90} h^5 f^{(4)}(x_0)$$

$\therefore$  by neglecting higher terms.

## (III) Error Analysis of Simpson's $\frac{3}{8}$ Rule

$n=3$ .

$$E = X - A$$

$$= \int_{x_0}^{x_3} f(x) dx - \frac{3h}{8} \left\{ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right\}$$

$$= I_1 - I_2 \rightarrow (ii)$$

$$I_1 = \int_{x_0}^{x_3} f(x) dx \quad \text{Let } f(x) = F'(x)$$

$$= \int_{x_0}^{x_3} F'(x) dx$$

$$= [F(x)]_{x_0}^{x_3} = F(x_3) - F(x_0)$$

$$= F(x_0 + 3h) - F(x_0)$$

$$= F(x_0) + 3h F'(x_0) + \frac{(3h)^2}{2!} F''(x_0) + \frac{(3h)^3}{3!} F'''(x_0) + \frac{(3h)^4}{4!} F^{(4)}(x_0) + \dots - F(x_0)$$

$$= \frac{(3h)^3}{3!} F'''(x_0) + \frac{(3h)^4}{4!} F^{(4)}(x_0) + \dots - F(x_0)$$



$$I_1 = 3h f(x_0) + \frac{9h^2}{2} f'(x_0) + \frac{9h^3}{2} f''(x_0) + \frac{27h^4}{8} f'''(x_0) + \frac{31}{40} h^5 f^{(iv)}(x_0) + \dots$$

And

$$I_2 = \frac{3h}{8} [f(x_0) + 3f(x_0+h) + 3f(x_0+2h) + f(x_0+3h)]$$

$$= \frac{3h}{8} \left[ f(x_0) + 3 \left\{ f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(iv)}(x_0) + \dots \right\} \right.$$

$$+ 3 \left\{ f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!} f''(x_0) + \frac{(2h)^3}{3!} f'''(x_0) + \frac{(2h)^4}{4!} f^{(iv)}(x_0) + \dots \right\} +$$

$$\left. f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2!} f''(x_0) + \frac{(3h)^3}{3!} f'''(x_0) + \frac{(3h)^4}{4!} f^{(iv)}(x_0) + \frac{(3h)^5}{5!} f^{(v)}(x_0) + \dots \right]$$

$$= 3hf(x_0) + \frac{9h^2}{2} f'(x_0) + \frac{9h^3}{2} f''(x_0) + \frac{27h^4}{8} f'''(x_0) + \frac{33h^5}{16} f^{(iv)}(x_0) + \dots$$

Put in (ii)

$$E = I_1 - I_2$$

$$= -\frac{3}{80} h^5 f^{(iv)}(x_0) + \dots$$

$$= -\frac{3}{80} h^5 f^{(iv)}(x_0) \quad (\text{Required})$$

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#### IV Error Analysis of Boole's Rule :-

;  $n = 4$ 

$$E = X - A$$

$$= \int_{x_0}^{x_4} f(x) dx - \frac{2h}{45} \left\{ 7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right\}$$

$$= I_1 - I_2 \rightarrow (iii)$$

$$\text{Now } I_1 = \int_{x_0}^{x_4} f(x) dx$$

$$= F(x_4) - F(x_0) \quad \because f(x) = F'(x)$$

$$= F(x_0 + 4h) - F(x_0)$$

$$= F(x_0) + 4hF'(x_0) + \frac{(4h)^2}{2!} F''(x_0) + \frac{(4h)^3}{3!} F'''(x_0) + \frac{(4h)^4}{4!} F^{(iv)}(x_0) + \frac{(4h)^5}{5!} F^{(v)}(x_0) + \dots - F(x_0)$$

$$= 4hf(x_0) + 8h^2 f'(x_0) + \frac{32h^3}{3} f''(x_0) +$$

$$\frac{32h^4}{3} f'''(x_0) + \frac{128h^5}{15} f^{(iv)}(x_0) + \frac{256h^6}{45} f^{(v)}(x_0) + \dots$$

$$+ \frac{1024h^7}{315} f^{(vi)}(x_0) + \dots$$

$$\text{Now } I_2 = \frac{2h}{45} \left\{ 7f(x_0) + 32f(x_0+h) + 12f(x_0+2h) + 32f(x_0+3h) + 7f(x_0+4h) \right\}$$

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$$I_2 = \frac{2h}{45} \left[ 7f(x_0) + 32 \left\{ \frac{f(x_0) + hf'(x_0) + h^2 f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(4)}(x_0) + \dots \right\} \right. \\
+ 12 \left\{ \frac{f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!} f''(x_0) + \frac{(2h)^3}{3!} f'''(x_0) + \frac{(2h)^4}{4!} f^{(4)}(x_0) + \frac{(2h)^5}{5!} f^{(5)}(x_0) + \frac{(2h)^6}{6!} f^{(6)}(x_0) + \dots \right\} \\
+ 32 \left\{ \frac{f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2!} f''(x_0) + \frac{(3h)^3}{3!} f'''(x_0) + \frac{(3h)^4}{4!} f^{(4)}(x_0) + \frac{(3h)^5}{5!} f^{(5)}(x_0) + \frac{(3h)^6}{6!} f^{(6)}(x_0) + \dots \right\} \\
+ 7 \left\{ \frac{f(x_0) + 4hf'(x_0) + \frac{(4h)^2}{2!} f''(x_0) + \frac{(4h)^3}{3!} f'''(x_0) + \frac{(4h)^4}{4!} f^{(4)}(x_0) + \frac{(4h)^5}{5!} f^{(5)}(x_0) + \frac{(4h)^6}{6!} f^{(6)}(x_0) + \dots \right\} \left. \right]$$

$$= \frac{4h}{3} f(x_0) + 8h^2 f'(x_0) + \frac{32}{15} h^3 f''(x_0) + \frac{32h^4}{3} f'''(x_0) + \frac{128}{15} h^5 f^{(4)}(x_0) + \frac{256}{45} h^6 f^{(5)}(x_0) \\
+ \frac{88h^7}{27} f^{(6)}(x_0) + \frac{1552}{945} h^8 f^{(7)}(x_0) + \dots$$

Put in (iii)

$$E = \frac{4h}{3} f(x_0) + 8h^2 f'(x_0) + \frac{32}{15} h^3 f''(x_0) + \frac{32h^4}{3} f'''(x_0) + \frac{128}{15} h^5 f^{(4)}(x_0) + \frac{256}{45} h^6 f^{(5)}(x_0) \\
+ \frac{1024}{315} h^7 f^{(6)}(x_0) + \frac{512}{315} h^8 f^{(7)}(x_0) + \dots - \\
\left[ \frac{4h}{3} f(x_0) - 8h^2 f'(x_0) - \frac{32}{3} h^3 f''(x_0) - \frac{32}{3} h^4 f'''(x_0) \right. \\
\left. - \frac{128}{15} h^5 f^{(4)}(x_0) - \frac{256}{45} h^6 f^{(5)}(x_0) - \frac{88}{27} h^7 f^{(6)}(x_0) - \dots \right]$$

Neglecting  $h^3$  and higher terms, we get

$$E = \frac{-8h^7}{945} f^{(7)}(x_0)$$

(Required).

(v) Error Analysis of Weddle's Rule;  
 $n=6$ 

$$E = X_{x_6} - A \\
= \int_{x_0}^{x_6} f(x) dx - \frac{3h}{10} \left\{ f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6) \right\}$$

$$= I_1 - I_2 \rightarrow (iv)$$

$$I_1 = \int_{x_0}^{x_6} f(x) dx \quad (F'(x) = f(x)) \\
= F(x_6) - F(x_0) \\
= F(x_0 + 6h) - F(x_0) \\
= F(x_0) + 6h F'(x_0) + \frac{(6h)^2}{2!} F''(x_0) + \frac{(6h)^3}{3!} F'''(x_0) \\
+ \frac{(6h)^4}{4!} F^{(4)}(x_0) + \frac{(6h)^5}{5!} F^{(5)}(x_0) + \frac{(6h)^6}{6!} F^{(6)}(x_0) \\
+ \frac{(6h)^7}{7!} F^{(7)}(x_0) + \dots - F(x_0)$$

$$I_1 = 6h f(x_0) + 18h^2 f'(x_0) + 36h^3 f''(x_0) + 54h^4 f'''(x_0) + \frac{324}{5} h^5 f^{(4)}(x_0) + \frac{324}{5} h^6 f^{(5)}(x_0) \\
+ \frac{1944}{35} h^7 f^{(6)}(x_0) + \frac{1458}{35} h^8 f^{(7)}(x_0) + \dots$$

$$I_2 = \frac{3h}{10} \left[ f(x_0) + 5f(x_0+h) + f(x_0+2h) + 6f(x_0+3h) + f(x_0+4h) + 5f(x_0+5h) + f(x_0+6h) \right] \rightarrow *$$

By Taylor's Series find  $f(x_0+nh)$   
 $n = 1, 2, 3, 4, 5, 6$

then put in \* we get

$$= \frac{3h}{10} \left\{ 20f(x_0) + 60hf'(x_0) + 120h^2f''(x_0) + 180h^3f'''(x_0) + 216h^4f^{(4)}(x_0) + 216h^5f^{(5)}(x_0) + \frac{111}{6}h^6f^{(6)}(x_0) + \frac{1945}{7}h^7f^{(7)}(x_0) + \frac{77873}{840}h^8f^{(8)}(x_0) + \dots \right\}$$

$$= 6hf(x_0) + 18h^2f'(x_0) + 36h^3f''(x_0) + 54h^4f'''(x_0) + \frac{324}{5}h^5f^{(4)}(x_0) + \frac{324}{5}h^6f^{(5)}(x_0) + \frac{111}{20}h^7f^{(6)}(x_0) + \frac{1167}{34}h^8f^{(7)}(x_0) + \dots$$

put in (iv)

$$E = 6hf(x_0) + 18h^2f'(x_0) + 36h^3f''(x_0) + 54h^4f'''(x_0) + \frac{324}{5}h^5f^{(4)}(x_0) + \frac{324}{5}h^6f^{(5)}(x_0) + \frac{1944}{35}h^7f^{(6)}(x_0) + \frac{1458}{35}h^8f^{(7)}(x_0) + \dots - 6hf(x_0) - 18h^2f'(x_0) - 36h^3f''(x_0) -$$

$$54h^4f'''(x_0) - \frac{324}{5}h^5f^{(4)}(x_0) - \frac{324}{5}h^6f^{(5)}(x_0) -$$

$$\frac{111}{20}h^7f^{(6)}(x_0) - \frac{1167}{34}h^8f^{(7)}(x_0) - \dots$$

$$= -\frac{1}{140}h^7f^{(7)}(x_0) + \frac{8727}{1190}h^8f^{(8)}(x_0) + \dots$$

$$= -\frac{1}{140}h^7f^{(7)}(x_0)$$

by neglecting  $h^8$  & higher power.

which is required error.

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## DIFFERENCE EQUATION

A difference equation is a relation between the differences of an unknown function at one or more general values of the arguments.

For example:

$$\begin{aligned} y_{n+2} + 2y_{n+1} + y_n &= n^2 \\ \Delta^2 y_n + \Delta y_n + y_n &= n^2 \\ (E^2 + 2E + 1)y_n &= n^2 \end{aligned}$$

### \* Order of Difference Equation:

The order of difference equation is the difference between largest and smallest argument (suffix) in equation.

For example:

$$y_{k+2} + 2y_{k+1} + y_k = k^2$$

order is  $k+2 - k = 2$ .

### \* FORMATION OF DIFFERENCE EQUATION :-

The following example illustrates the way in which difference equation is formed.

### Example (i)

From  $y_n = A2^n + B(-3)^n$ . Derive difference equation not containing A and B.

Solution:-

$$\begin{aligned} y_n &= A2^n + B(-3)^n \rightarrow (i) \\ y_{n+1} &= A(2)^{n+1} + B(-3)^{n+1} \rightarrow (ii) \\ y_{n+2} &= A(2)^{n+2} + B(-3)^{n+2} \rightarrow (iii) \end{aligned}$$

From (i), (ii) and (iii)

$$\begin{aligned} y_n &= A2^n + B(-3)^n \\ y_{n+1} &= 2A2^n + (-3)B(-3)^n \\ y_{n+2} &= 4A2^n + 9B(-3)^n \end{aligned}$$

Eliminate  $A2^n$  and  $B(-3)^n$  from above equations.

|           |   |    |     |
|-----------|---|----|-----|
| $y_n$     | 1 | 1  | = 0 |
| $y_{n+1}$ | 2 | -3 |     |
| $y_{n+2}$ | 4 | 9  |     |

$$= y_n [18 + 12] - 1 [9y_{n+1} + 3y_{n+2}] + 1 [4y_{n+1} - 2y_{n+2}]$$

$$= y_{n+2} + y_{n+1} - 6y_n = 0$$

which is required difference equation.

### Example (ii)

Eliminate constant from the equation  $y_n = A_1 + (-1)^n A_2 + A_3 n$  and derive corresponding difference equation.



Solution :-

$$y_n = A_1 + (-1)^n A_2 + A_3 n \rightarrow (i)$$

$$y_{n+1} = A_1 + (-1)^{n+1} A_2 + A_3 (n+1)$$

$$y_{n+1} = A_1 + (-1)(-1)^n A_2 + A_3 n + A_3 \rightarrow (ii)$$

$$y_{n+2} = A_1 + (-1)^{n+2} A_2 + A_3 (n+2) \rightarrow (iii)$$

$$y_{n+2} = A_1 + (-1)^n A_2 + n A_3 + 2 A_3 \rightarrow (iv)$$

$$y_{n+3} = A_1 + (-1)^{n+3} A_2 + n A_3 + 3 A_3 \rightarrow (v)$$

Subtract (i) from (iii)

$$y_{n+2} - y_n = 2 A_3 \rightarrow (vi)$$

Subtract (ii) from (iv)

$$y_{n+3} - y_{n+1} = 2 A_3 \rightarrow (vii)$$

From (vi) and (vii)

$$y_{n+3} - y_{n+1} = y_{n+2} - y_n$$

$$\Rightarrow y_{n+3} - y_{n+2} = y_{n+1} - y_n = a.$$

is required difference equation. (Ans)

Example (iii):

$$\text{Also for } y_n = (A + Bn) 3^n$$

the difference equation is

$$y_{n+2} - 6y_{n+1} + 9y_n = 0. \quad (\text{Ans})$$

## Linear Difference Equations:

LDE is that in which  $y_n, y_{n+1}, y_{n+2}$  etc occur to first degree only and are not multiplied.

A Linear difference equation with constant coefficient is of the form  $a_0 y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_n y_n = \phi(n)$

The equation can be written symbolically as

$$f(E) y_n = \phi(n)$$

$$y_n \phi(n) = 0 \quad \text{homogeneous} \rightarrow (i)$$

$$y_n \phi(n) \neq 0 \quad \text{Non homogeneous} \rightarrow (ii)$$

1<sup>st</sup> We Solve associated homogeneous

$$f(E) y_n = 0$$

The solution of that equation is called Complementary function (C.F). It contains arbitrary constant equal to order of that equation.

2<sup>nd</sup> solve associated Non-homogeneous

$$f(E) y_n = \phi(n)$$

The solution is called particular solution, which does not contain arbitrary constant.

General solution of  $f(E) y_n = \phi(n)$  is  $y_n = C.F + P.S.$

§ Rule for finding Complementary Function (C.F.) of equation  $f(E)y_n = \phi(n)$ .

Solve  $f(E) = 0$  which is called Auxiliary equation.

Let  $\alpha_1, \alpha_2, \alpha_3, \dots$  be root of auxiliary equation

(I) If  $\alpha_1, \alpha_2, \dots$  are all real and distinct Then C.F. is

$$y_n = C_1 \alpha_1^n + C_2 \alpha_2^n + C_3 \alpha_3^n + \dots$$

(II) If two roots are equal  $\alpha_1 = \alpha_2 = \alpha$  Then C.F. is

$$y_n = (C_1 + C_2 n) \alpha^n + C_3 \alpha_3^n + \dots$$

Similarly if three roots are equal

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$$

$$y_n = (C_1 + C_2 n + C_3 n^2) \alpha^n + C_4 \alpha_4^n + \dots$$

(III) If two roots are imaginary say  $\alpha \pm i\beta$  then C.F. is

$$y_n = R^n (C_1 \cos n\theta + C_2 \sin n\theta) + C_3 \alpha_3^n + \dots$$

Where

$$R = \sqrt{\alpha^2 + \beta^2}$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Questions:- Solve

$$(i) y_{k+2} - 3y_{k+1} + 2y_k = 0$$

Equation in E-operator form is

$$E^2 y_k - 3E y_k + 2y_k = 0$$

$$(E^2 - 3E + 2)y_k = 0$$

The auxiliary equation is

$$E^2 - 3E + 2 = 0$$

$$(E-1)(E-2) = 0$$

$$\Rightarrow E = 1, 2$$

$$y_k = C_1 (1)^n + C_2 (2)^n \quad (\text{Answer})$$

$$(ii) U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$$

$$E^3 U_n - 2E^2 U_n - 5E U_n + 6U_n = 0$$

$$(E^3 - 2E^2 - 5E + 6)U_n = 0$$

Auxiliary equation is

$$E^3 - 2E^2 - 5E + 6 = 0$$

$$(E-1)(E^2 - E - 6) = 0$$

$$(E-1)(E+2)(E+3) = 0$$

$$\Rightarrow E = 1, -2, +3$$

$$U_n = C_1 (1)^n + C_2 (-2)^n + C_3 (+3)^n$$

(iii) Find C.F. of  $(E^2 + 2E + 4)y_k = 0$ .

$$E^2 + 2E + 4 = 0$$

$$E = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{2} i$$

$$= -1 \pm \sqrt{3} i$$



$$a = -1, b = +\sqrt{3}$$

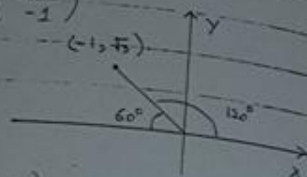
$$R = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3}$$

$$= \sqrt{4} = 2$$

$$\theta_1 = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{+\sqrt{3}}{-1}\right)$$

$$\theta_1 = 120^\circ = \frac{2\pi}{3}$$



Solution is

$$y_k = R^k (C_1 \cos k\theta_1 + C_2 \sin k\theta_1)$$

$$= 2^k \left( C_1 \cos \frac{2\pi}{3} k + C_2 \sin \frac{2\pi}{3} k \right)$$

(Ans.)

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### (Exercise)

Solve following difference equations:-

(i)  $U_{n+2} - 2U_{n+1} + U_n = 0$

Ans:  $U_n = C_1 + C_2 n$

(ii)  $y_{n+3} - 3y_{n+2} + 4y_n = 0$

Ans:  $y_n = C_1(-1)^n + (C_2 + C_3 n) 2^n$

(iii)  $(E^2 + 2E + 2)f_n = 0$

Ans:  $2^{n/2} \left[ \cos \frac{3\pi n}{4} + B \sin \frac{3\pi n}{4} \right]$

(iv)  $y_{k+4} - 2y_{k+3} + 2y_{k+2} + (-2)y_{k+1} + y_k = 0$

Ans:

$$y_k = (A + Bk) + \left( C \cos \frac{\pi}{2} k + D \sin \frac{\pi}{2} k \right)$$

(v)  $y_{n+2} - y_{n+1} + y_n = 0$  ;  $y_0 = 1$

$$y_1 = 1 + \frac{\sqrt{3}}{2}$$

Ans:  $y_n = \cos \frac{\pi}{3} n + \sin \frac{\pi}{3} n$

(vi)  $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$

Ans:  $y_k = C_1(-1)^k + C_2(-2)^k + C_3(-3)^k$

(vii)  $y_{t+2} + \frac{1}{4} y_t = 0$  ;  $y_0 = 1, y_1 = 1$

Ans:  $y_t = \left(\frac{1}{2}\right)^t \left( \cos \frac{\pi}{2} t + 2 \sin \frac{\pi}{2} t \right)$

(viii)  $\Delta^3 y_n - 5\Delta y_n + 4y_n = 0$

Ans:  $y_n = C_1 2^n + C_2 \left( \frac{1+\sqrt{17}}{2} \right)^n + C_3 \left( \frac{1-\sqrt{17}}{2} \right)^n$

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(ix)  $y_{m+2} + 16y_{m-1} = 0$

Ans:

$$y_m = 2^m \left[ C_1 \cos \frac{\pi}{4} m + C_2 \sin \frac{\pi}{4} m + C_3 \cos \frac{3\pi}{4} m + C_4 \sin \frac{3\pi}{4} m \right]$$

(x) Let  $U_k$  and  $V_k$  be solution of  
 $\rightarrow * y_{k+2} + a_1 y_{k+1} + a_2 y_k = 0$  then show that  
 $C_1 U_k + C_2 V_k$  is also solution.

Sol:  $U_k$  and  $V_k$  be solution then

$$U_{k+2} + a_1 U_{k+1} + a_2 U_k = 0 \rightarrow (i)$$

$$V_{k+2} + a_1 V_{k+1} + a_2 V_k = 0 \rightarrow (ii)$$

We show that  $C_1 U_k + C_2 V_k$  is also solution.  
 xing (i) by  $C_1$  and (ii) by  $C_2$  then  
 add

$$(C_1 U_{k+2} + C_2 V_{k+2}) + a_1 (C_1 U_{k+1} + C_2 V_{k+1}) + a_2 (C_1 U_k + C_2 V_k) = 0$$

Comparing with \* it follows that  
 $C_1 U_k + C_2 V_k$  is also solution.

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## Solution of Non-homogeneous Linear eq. $f(E)y_n = \phi(n)$

### Type (1):

When  $\phi(n) = \text{Constant}$

The trial substitution for particular solution is  
 $y_n = C$

(i) If sum of coeff. of Auxiliary equation is  
 zero then  $y_n = C$  fails.

In this case  $y_n = C$  (some power of  $n$ ).

(ii) If sum of coeff. of Auxiliary equation is  
 zero then '1' be root of equation.

(iii) If 1 is single root put  $y_n = Cn$

If 1 is double root put  $y_n = Cn^2$

If 1 is triple root put  $y_n = Cn^3$

..... So on.

\* Question: Solve

$$y_{k+3} + 3y_{k+2} - 2y_k = 5 \rightarrow (i)$$

Solution:  $E^3 y_k + 3E^2 y_k - 2y_k = 5$

$$(E^3 + 3E^2 - 2)y_k = 5$$

Auxiliary equation is

$$E^3 + 3E^2 - 2 = 0$$

$$E = -1 \Rightarrow E^3 + 3E^2 - 2 = 0$$

$$(-1)^3 + 3(-1)^2 - 2 = 0$$

$$-1 + 3 - 2 = 0$$

So  $E = -1$  is root.

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$$\begin{array}{r|rrrr} \text{by} & -1 & 1 & 3 & 0 & -2 \\ & & & -1 & -2 & +2 \\ & 1 & 2 & -2 & 0 & \end{array}$$

$$\Rightarrow E^2 + 2E - 2 = 0$$

$$\Rightarrow E = -1 \pm \sqrt{3}$$

$$\Rightarrow \text{Roots are } -1, -1 \pm \sqrt{3}$$

Complement function is

$$y_k = A(-1)^k + B(-1 + \sqrt{3})^k + C(-1 - \sqrt{3})^k$$

and particular solution put  $y_k = C$

$$y_k = C$$

$$y_{k+1} = C$$

$$y_{k+2} = C$$

$$y_{k+3} = C$$

put in (i),

$$C + 3C - 2C = 5$$

$$2C = 5$$

$$C = \frac{5}{2}$$

$$y_k = 5/2$$

$$\text{P.S.} = y_k = 5/2$$

$$\text{General sol} = \text{C.F.} + \text{P.S.}$$

$$= A(-1)^k + B(-1 + \sqrt{3})^k + C(-1 - \sqrt{3})^k + 5/2$$

(Answer)

Question:

$$y_{t+2} + \frac{1}{4}y_t = 2; y_0 = y_1 = 1$$

Solution:

$$y_{t+2} + \frac{1}{4}y_t = 2$$

$$\Rightarrow E^2 y_t + \frac{1}{4}y_t = 2$$

$$\Rightarrow (E^2 + \frac{1}{4})y_t = 2$$

$$\text{Ch. eq. is } E^2 + \frac{1}{4} = 0 \Rightarrow E = \pm \frac{1}{2}i$$

$$\text{C.F. equation is } y_t = R^t [A \cos t\theta + B \sin t\theta]$$

where

$$R = \sqrt{a^2 + b^2} = \sqrt{0 + (\frac{1}{2})^2} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\frac{1}{2}}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$y_t = \left(\frac{1}{2}\right)^t \left[ A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t \right]$$

$$\text{For P.S. put } y_k = C$$

$$y_{t+1} = y_{t+2} = C$$

$$\text{then } C + \frac{1}{4}C = 2$$

$$\frac{5}{4}C = 2 \Rightarrow C = \frac{8}{5}$$

$$y_t = \frac{8}{5}$$

$$\text{G.S.} = \text{C.F.} + \text{P.S.}$$

$$= \left(\frac{1}{2}\right)^t \left[ A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t \right] + \frac{8}{5}$$

Also find A and B Give  $y_0 = y_1 = 1$

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When  $t=0$ 

$$y_0 = \left(\frac{1}{2}\right)^0 [A \cos(0) + B \sin(0)] + \frac{8}{5}$$

$$1 = A + \frac{8}{5}$$

$$\Rightarrow A = -\frac{3}{5}$$

When  $t=1$ 

$$y_1 = \left(\frac{1}{2}\right)^1 [A \cos\left(\frac{\pi}{2}\right) + B \sin\left(\frac{\pi}{2}\right)] + \frac{8}{5}$$

$$1 = \frac{1}{2} \left[ -\frac{3}{5}(0) + B(1) \right] + \frac{8}{5}$$

$$1 = \frac{B}{2} + \frac{8}{5}$$

$$\Rightarrow B = 2\left(1 - \frac{8}{5}\right) = -\frac{6}{5}$$

Then

$$y_t = \left(\frac{1}{2}\right)^t \left[ -\frac{3}{5} \cos \frac{\pi}{2} t - \frac{6}{5} \sin \frac{\pi}{2} t \right] + \frac{8}{5}$$

(Answer).

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## (Exercise)

Solve

$$1) y_{n+2} - 4y_{n+1} + y_n = 1$$

$$\text{Ans: } y_n = C_1 (2 + \sqrt{3})^n + C_2 (2 - \sqrt{3})^n - \frac{1}{2}$$

$$2) y_{n+2} - 3y_{n+1} - 4y_n = 6$$

$$\text{Ans: } y_n = C_1 (-1)^n + C_2 4^n + 1$$

$$3) u_{n+2} - 4u_{n+1} + u_n = 3$$

$$\text{Ans: } u_n = C_1 (2 + \sqrt{3})^n + C_2 (2 - \sqrt{3})^n - \frac{3}{2}$$

$$4) y_{n+3} + 6y_{n+2} + 11y_{n+1} = 5$$

$$5) \Delta^2 y_k + 3\Delta y_k = 9; \quad y_0 = 1, y_1 = 0$$

$$6) (E^3 - 4E^2 + 2E + 1)y_n = 14$$

$$\text{Ans: } y_n = C_1 (+1)^n + C_2 \left( \frac{3 + \sqrt{13}}{2} \right)^n +$$

$$C_3 \left( \frac{3 - \sqrt{13}}{2} \right)^n - \frac{14}{3} n$$



Type (2):When  $\phi(n) = \alpha a^n$ Where  $\alpha$  is constant.

For P.S. trial substitution is

 $y_n = C a^n$  where  $a$  is not root.  
When  $a$  is root of auxiliary equation,  
trial substitution is given below.(i)  $a$  is single root put  $y_n = C n a^n$ (ii)  $a$  is double root put  $y_n = C n^2 a^n$ (iii)  $a$  is triple root put  $y_n = C n^3 a^n$ 

So on.

Example :-

Solve  $U_{n+2} - 7U_{n+1} + 10U_n = 12(4)^n$   
 $\rightarrow (i)$

Solution :-  $U_{n+2} - 7U_{n+1} + 10U_n = 12(4)^n$

$E^2 U_n - 7E U_n + 10U_n = 12(4)^n$

$(E^2 - 7E + 10)U_n = 12(4)^n$  For C.F

Ch. equation  $E^2 - 7E + 10 = 0$

$E^2 - 2E - 5E + 10 = 0$

$E(E-2) - 5(E-2) = 0$

$(E-2)(E-5) = 0$

$\Rightarrow E = +2, 5$

C.F is  $U_n = A(2)^n + B(5)^n$

For P.S

put  $U_n = C 4^n$  Here  $a = 4$

$U_{n+1} = C(4)^{n+1}$

$U_{n+2} = C(4)^{n+2}$

Put in (i)

$C 4^{n+2} - 7C 4^{n+1} + 10C 4^n = 12 4^n$

$4^n [C 4^2 - 7C 4^1 + 10C] = 12 4^n$

$\Rightarrow 16C - 28C + 10C = 12$

$\Rightarrow -2C = 12$

$\Rightarrow C = -6$

$\Rightarrow U_n = C 4^n$   
 $= -6(4)^n$

General solution = C.F + P.S

$U_n = A(2)^n + B(5)^n - 6 4^n$

(Answer)

Example:

Solve

$y_{k+2} - 4y_{k+1} + 4y_k = 2^{k+1} \rightarrow (i)$

Solution:

$y_{k+2} - 4y_{k+1} + 4y_k = 2 \cdot 2^k$

$E^2 y_k - 4E y_k + 4y_k = 2(2)^k$

$\Rightarrow (E^2 - 4E + 4)y_k = 2(2)^k$

For C.F

Ch. eq.  $E^2 - 4E + 4 = 0$

$E^2 - 2E - 2E + 4 = 0$

$E(E-2) - 2(E-2) = 0$

$\Rightarrow (E-2)(E-2) = 0$

$\Rightarrow E = 2, 2$

$y_k = (A + Bk)(2)^k$

For Particular Solution.

put  $y_k^* = C k^2 2^k$

( $\because 2$  is double root)

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$$y_{k+1} = C(k+1)^2 (2)^{k+1}$$

$$y_{k+2} = C(k+2)^2 (2)^{k+2}$$

put in (i)

$$\begin{aligned} C(k+2)^2 (2)^{k+2} - 4C(k+1)^2 (2)^{k+1} + 4Ck^2 (2)^k &= 2 \cdot 2^k \\ \Rightarrow [C(k^2 + 4 + 4k) 2^{k+2} - 4C(k^2 + 1 + 2k) 2^{k+1} + 4Ck^2] 2^k &= 2 \cdot 2^k \\ &= 2 \cdot 2^k \end{aligned}$$

$$\begin{aligned} \Rightarrow C[(k^2 + 4 + 4k)4 - 4(2)(k^2 + 1 + 2k) + 4k^2] &= 2 \\ \Rightarrow C[4k^2 + 16 + 16k - 8k^2 - 8 - 16k + 4k^2] &= 2 \\ \Rightarrow C[8] &= 2 \end{aligned}$$

$$\Rightarrow C = \frac{1}{4}$$

$$y_k^* = \frac{1}{4} k^2 2^k$$

General solution is

$$C.F. + P.S$$

$$\begin{aligned} \Rightarrow y_k &= y_k + y_k^* \\ &= (A + Bk) 2^k + \frac{1}{4} k^2 2^k \end{aligned}$$

(Answer)

Example: Solve  $U_{n+2} - 7U_{n+1} + 10U_n = 12 \cdot 5^n$ 

Solution:

$$U_{n+2} - 7U_{n+1} + 10U_n = 12 \cdot 5^n$$

$$E^2 U_n - 7E U_n + 10U_n = 12(5)^n$$

$$(E^2 - 7E + 10)U_n = 12(5)^n$$

For C.F

$$\text{ch. eq. } E^2 - 7E + 10 = 0$$

$$E = 2, 5 \Rightarrow U_n = C_1 2^n + C_2 5^n$$

$$\text{For P.S put } U_n^* = Cn 5^n$$

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$$U_{n+1} = C(n+1)5^{n+1}$$

$$U_{n+2} = C(n+2)5^{n+2}$$

then  $U_{n+2} - 7U_{n+1} + 10U_n = 2 \cdot 5^n$  becomes

$$C(n+2)5^{n+2} - 7C(n+1)5^{n+1} + 10Cn5^n = 2 \cdot 5^n$$

$$5^n [C(n+2)5^2 - 7C(n+1)5 + 10Cn] = 2 \cdot 5^n$$

$$\Rightarrow C(n+2)5^2 - 35C(n+1) + 10Cn = 2$$

$$C[25n + 50 - 35n - 35 + 10n] = 2$$

$$C[25n + 50 - 35n - 35] = 2$$

$$C(-10n + 15) = 2$$

$$C = \frac{12}{15} = \frac{4}{5}$$

$$\Rightarrow U_n^* = \frac{4}{5} n 5^n$$

$$\text{General Sol} = C.F. + P.S$$

$$= U_n + U_n^*$$

$$= C_1 2^n + C_2 5^n + \frac{4}{5} n 5^n$$

(Answer)

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# EXERCISE

Solve:

1)  $y_{n+4} - A^4 y_n = 2^n$

Ans:  $y_n = C_1 A^n + C_2 (-A)^n +$

$$A^n \left( C_3 \cos \frac{\pi}{2} n + C_4 \sin \frac{\pi}{2} n \right) + \frac{1}{16 - A^4} 2^n$$

2)  $y_{n+2} + 4y_{n+1} + 4y_n = 2^{n+2}$ ;  $y_0 = 1, y_1 = 2$

Ans:  $y_n = \left(1 - \frac{n}{2}\right) 2^n + \frac{1}{2} n^2 2^n$

3)  $y_{k+2} - 6y_{k+1} + 8y_k = 2 \cdot 3^k$

Ans:  $y_k = C_1 2^k + C_2 4^k = 2 \cdot 3^k$

4)  $y_{k+3} - y_{k+2} + y_{k+1} - y_k = 3^k$

Ans:  $y_n = C_1 + C_2 \cos \frac{n\pi}{2} + C_3 \sin \frac{n\pi}{2} + \frac{1}{2} 3^k$

5)  $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 3^n$

Ans:  $y_n = C_1 + nC_2 + n^2 C_3 + 3^n$

6)  $y_{n+2} - 3y_{n+1} + 2y_n = 7 \cdot 2^n$

Ans:  $y_n = C_1 + C_2 n^2 + 7n 2^{n-1}$

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Near Moon CHG, Rwp  
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## Type (3):

When  $\phi(n) = \text{polynomial}$ Like  $\phi(n) = 1 + k^2$  or  $k^3 + k^2 + k + 1$ The trial substitution for P.S is  $y_n = B_0 + B_1 n + B_2 n^2 + \dots + B_r n^r$  (degree of polyn)

(i) If sum of Coeffs. in  $f(E)y_n = 0$  equation is zero then 1 is root of above equation.

then  $y_n = B_0 + B_1 n + B_2 n^2 + \dots + B_r n^r$  fails.

In this case multiply trial solution as below

(1) 1 is single put  $y_n = n(B_0 + B_1 n + \dots + B_r n^r)$

(2) 1 is double put  $y_n = n^2(\dots)$

(3) 1 is triple put  $y_n = n^3(\dots)$

..... So on.

Example: Solve

$$y_{n+2} - 2y_{n+1} + y_n = n^2$$

Solution:  $y_{n+2} - 2y_{n+1} + y_n = n^2$

$$E^2 y_n - 2E y_n + y_n = n^2$$

$$\Rightarrow (E^2 - 2E + 1) y_n = n^2$$

For Complementary function ...

Ch. eq  $E^2 - 2E + 1 = 0$

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$$\Rightarrow (E-1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

$$C.F. \Rightarrow y_n = (C_1 + C_2 n)(1)^n$$

$$y_n = C_1 + C_2 n$$

As 1 is double root so put For P.S

$$y_n^* = n^2 (B_0 + B_1 n + B_2 n^2)$$

$$y_n = n^2 B_0 + B_1 n^3 + B_2 n^4$$

$$y_{n+1} = (n+1)^2 B_0 + (n+1)^3 B_1 + (n+1)^4 B_2$$

$$y_{n+2} = (n+2)^2 B_0 + (n+2)^3 B_1 + (n+2)^4 B_2$$

then  $y_{n+2} - 2y_{n+1} + y_n = n^2$  becomes

$$B_0(n+2)^2 + B_1(n+2)^3 + B_2(n+2)^4 +$$

$$(-2)[B_0(n+1)^2 + B_1(n+1)^3 + B_2(n+1)^4] +$$

$$B_0 n^2 + B_1 n^3 + B_2 n^4 = n^4$$

$$\Rightarrow B_0(n^2 + 4n + 4) + B_1(n^3 + 6n^2 + 12n + 8) +$$

$$B_2(n^4 + 8n^3 + 24n^2 + 32n + 16)$$

$$- 2B_0(n^2 + 2n + 1) - 2B_1(n^3 + 3n^2 + 3n + 1)$$

$$- 2B_2(n^4 + 4n^3 + 6n^2 + 4n + 1) +$$

$$B_0 n^2 + B_1 n^3 + B_2 n^4 = n^2$$

Equating Coeff. of  $n^4$ :

$$B_2 - 2B_2 + B_2 = 0$$

Not gives information

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Equating Coeff. of  $n^3$ :

$$B_1 + 8B_2 - 2B_1 - 8B_2 + B_1 = 0$$

$$0 = 0$$

Equating Coeff. of  $n^2$ :

$$\cancel{B_0} + 6\cancel{B_1} + 24B_2 - 2\cancel{B_0} - 6\cancel{B_1} - 12B_2 + \cancel{B_0} = 1$$

$$12B_2 = 1$$

$$B_2 = \frac{1}{12}$$

Equating Coeff. of  $n$ :

$$4B_0 + 12B_1 + 32B_2 - 4B_0 - 6B_1 - 8B_2 = 0$$

$$6B_1 + 24B_2 = 0$$

$$B_1 = -4B_2$$

$$B_1 = -4\left(\frac{1}{12}\right) = -\frac{1}{3}$$

Equating Constant term:

$$4B_0 + 8B_1 + 16B_2 - 2B_0 - 2B_1 - 2B_2 = 0$$

$$2B_0 + 6B_1 + 14B_2 = 0$$

$$B_0 + 3B_1 + 7B_2 = 0$$

$$B_0 + 3\left(-\frac{1}{3}\right) + 7\left(\frac{1}{12}\right) = 0$$

$$B_0 - 1 + \frac{7}{12} = 0$$

$$\Rightarrow B_0 = \frac{5}{12}$$

$$\text{So } y_n^* = n^2 \left( \frac{5}{12} + \frac{-1}{3}n + \frac{1}{12}n^2 \right)$$

$$= \frac{5}{12}n^2 - \frac{1}{3}n^3 + \frac{1}{12}n^4$$

General solution is

$$\Rightarrow C.F + P.S$$

$$y_n = C_1 + C_2 n + \frac{5n^2}{12} - \frac{1n^3}{3} + \frac{1n^4}{12}$$

(Ans)

Example:

$$\text{Solve } \Delta^2 y_k + 3\Delta y_k + 3y_k = k^2 + 1 \rightarrow \textcircled{a}$$

$$\text{Solution: } \Delta^2 y_k + 3\Delta y_k + 3y_k = k^2 + 1$$

$$\text{As } \Delta y_k = y_{k+1} - y_k$$

$$\text{So } \Delta(\Delta y_k) + 3\Delta y_k + 3y_k = k^2 + 1$$

$$\Delta(y_{k+1} - y_k) + 3(y_{k+1} - y_k) + 3y_k = k^2 + 1$$

$$\Delta y_{k+1} - \Delta y_k + 3y_{k+1} - 3y_k + 3y_k = k^2 + 1$$

$$y_{k+2} - y_{k+1} - y_{k+1} + y_k + 3y_{k+1} = k^2 + 1$$

$$y_{k+2} - y_{k+1} + y_k = k^2 + 1 \rightarrow \textcircled{b}$$

For Complementary function:

$$\text{Ch. equation } y_{k+2} - y_{k+1} + y_k = 0$$

$$E^2 y_k - E y_k + y_k = 0$$

$$(E^2 - E + 1) y_k = 0, y_k \neq 0$$

$$\Rightarrow E^2 - E + 1 = 0$$

$$E = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

$$E = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$y_k = R^k [A_1 \cos k\theta + A_2 \sin k\theta]$$

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$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$y_k = (1)^k \left[ A_1 \cos \frac{\pi}{3} k + A_2 \sin \frac{\pi}{3} k \right] \rightarrow *$$

For Particular Solution

$$\text{Put } y_k^* = (B_0 + B_1 k + B_2 k^2)$$

then  $\textcircled{b}$  becomes

$$B_0 + B_1(k+2) + B_2(k+2)^2 - B_0 - B_1(k+1) - B_2(k+1) + B_0 + B_1 k + B_2 k^2 = k^2 + 1$$

$$B_0 + B_1(k+2) + B_2(k^2 + 4k + 4) - B_0 - B_1(k+1) - B_2(k^2 + 2k + 1) + B_0 + B_1 k + B_2 k^2 = k^2 + 1$$

Equating Coeff. of  $k^2$ :

$$B_2 - B_2 + B_2 = 1$$

$$\Rightarrow B_2 = 1$$

Equating Coeff. of  $k$ :

$$B_1 + 4B_2 - B_1 - 2B_2 + B_1 = 0$$

$$B_1 + 2B_2 = 0$$

$$B_1 + 2(1) = 0$$



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$$\Rightarrow B_1 = -2$$

Equating Constant term :-

$$B_0 + 2B_1 + 4B_2 - B_0 - B_1 - B_2 + B_0 = 1$$

$$B_0 + B_1 + 3B_2 = 1$$

$$B_0 + (-2) + 3(1) = 1$$

$$B_0 + 1 = 1$$

$$B_0 = 0$$

So  $y_k^*$  becomes

$$y_k^* = B_0 + B_1 k + B_2 k^2$$

$$y_k^* = -2k + k^2 \rightarrow *$$

General Solution = C.F + P.S

$$y_k' = y_k + y_k^*$$

From \* and \*

$$y_k' = (1)^k \left[ A_1 \cos \pi k + A_2 \sin \pi k \right] + \underset{3}{-2k} + \underset{3}{k^2}$$

(Ans)

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Example :

Solve

$$y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = k^2 \rightarrow (1)$$

Solution :

$$y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = k^2$$

$$E^4 y_k - 2E^3 y_k + 2E^2 y_k - 2E y_k + y_k = k^2$$

$$(E^4 - 2E^3 + 2E^2 - 2E + 1) y_k = k^2$$

For C.F

$$E^4 - 2E^3 + 2E^2 - 2E + 1 = 0$$

by Synthetic Division.

|   |   |    |    |    |    |
|---|---|----|----|----|----|
| 1 | 1 | -2 | 2  | -2 | 1  |
|   |   | 1  | -1 | +1 | -1 |
| 1 | 1 | -1 | 1  | -1 | 0  |
|   |   | 1  | 0  | 1  |    |
|   | 1 | 0  | 1  | 0  |    |

$$(E-1)(E-1)(E^2+1) = 0$$

$$\Rightarrow E = 1, 1$$

$$E = \sqrt{-1}$$

$$\Rightarrow E = \pm i$$

So roots are

$$E = 1, 1, \pm i$$

Complementary Solution is

$$y_k = (A+Bk)(1)^k + R^k [A_3 \cos k\theta + A_4 \sin k\theta]$$

$$R = \sqrt{0 + (1)^2} = 1$$

$$\theta = \tan^{-1} \left( \frac{1}{0} \right) = \frac{\pi}{2}$$

So

$$y_k = (A+Bk)(1)^k + (1)^k \left[ A_3 \cos \frac{\pi k}{2} + A_4 \sin \frac{\pi k}{2} \right]$$

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For P.S put  $y_k = k^2(B_0 + B_1 k + B_2 k^2)$   
in (i)

$$\begin{aligned} & (k+4)^2 [B_0 + B_1(k+4) + B_2(k+4)^2] \\ & - 2(k+3)^2 [B_0 + B_1(k+3) + B_2(k+3)^2] \\ & + 2(k+2)^2 [B_0 + B_1(k+2) + B_2(k+2)^2] \\ & - 2(k+1)^2 [B_0 + B_1(k+1) + B_2(k+1)^2] \\ & + k^2 [B_0 + B_1 k + B_2 k^2] = k^2 \end{aligned}$$

Comparing co-efficients of like power of  $n$  on both sides

$$\begin{aligned} B_0 + 12B_1 + 9B_2 - 2B_1 - 18B_1 - 108B_2 + 2B_0 \\ + 12B_1 + 48B_2 - 2B_0 - 6B_1 - 12B_2 + B_0 = 1 \end{aligned}$$

$$\begin{aligned} 8B_0 + 96B_1 + 256B_2 - 12B_0 - 54B_1 - 216B_2 + 8B_0 \\ + 24B_1 + 16B_2 - 4B_0 - 6B_1 - 8B_2 = 0 \end{aligned} \rightarrow *$$

$$\begin{aligned} 16B_0 + 64B_1 + 256B_2 - 18B_0 - 54B_1 - 162B_2 + 8B_0 \\ + 16B_1 + 32B_2 - 2B_0 - 2B_1 - 2B_2 = 0 \end{aligned} \rightarrow **$$

From \*, \*' and \*\* we have

$$B_1 = \frac{4}{315}$$

$$B_0 = \frac{131}{315}$$

$$\text{and } B_2 = -\frac{1}{63}$$

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So particular solution becomes

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$$y_k = k^2 \left( \frac{131}{315} + \frac{4}{315} k - \frac{1}{63} k^2 \right) \rightarrow **$$

From \*\* and \*\*' general solution is

$$\begin{aligned} y_k = & (A_1 + Bk)(1)^k + \left[ \frac{A_3 \cos \frac{\pi k}{2} + A_4 \sin \frac{\pi k}{2}}{2} \right] (1)^k \\ & + k^2 \left( \frac{131}{315} + \frac{4}{315} k - \frac{1}{63} k^2 \right) \end{aligned}$$

(Answer)

Exercise:

$$\begin{aligned} 1) \Delta^2 y_n - 7\Delta y_n - 6y_n &= 2^n + n^2 + 1 \\ \text{Ans: } C_1(-4)^n + C_2(13)^n - \frac{1}{1800} n^3 &+ \frac{1}{30} n^2 - \frac{7}{60} n + \frac{1}{1800} \end{aligned}$$

$$\begin{aligned} 2) y_{n+2} - 7y_{n+1} + 12y_n &= 12n + 8 \\ \text{Ans: } y_n &= 32(3)^n - 22(4)^n + 3 + 2n \end{aligned}$$

$$\begin{aligned} 3) \Delta^4 y_n &= n \\ \text{Ans: } y_n &= C_1 + C_2 n + C_3 n^2 + C_4 n^3 - \frac{1}{12} n^4 + \frac{1}{120} n^5 \end{aligned}$$

$$\begin{aligned} 4) y_{k+2} - 2y_{k+1} + y_k &= k+1 \\ \text{Ans: } y_k &= (C_1 + C_2 k) + \frac{1}{6} k^3 \end{aligned}$$

$$\begin{aligned} 5) y_{n+2} - 5y_{n+1} + 6y_n &= 2n + 1 + 2^n \\ \text{Ans: } y_n &= (C_1 - \frac{1}{2} n) 2^n + C_2 3^n + n + 2 \end{aligned}$$

$$\begin{aligned} 6) \Delta^2 y_k + 3\Delta y_k + 3y_k &= k^2 + 1 \\ \text{Ans: } y_k &= \frac{C_1 \cos k\pi}{3} + \frac{C_2 \sin k\pi}{3} + \frac{k^2 - 2k + 2}{3} \end{aligned}$$

Type (4):When  $\phi(n) = a^n F(n)$ where  $F(n)$  is polynomial  
of degree  $t$ .

For P.S. trial substitution

$$y_n = a^n (B_0 + B_1 n + B_2 n^2 + \dots + B_t n^t)$$

If 'a' is root of auxiliary equation then  
above substitution fails.In this case multiply the trial  
function by suitable power of  $n$ .If 'a' is single root then multiply by  $n$ If 'a' is double root then multiply by  $n^2$ If 'a' is triple root then multiply by  $n^3$ 

..... so on.

Example: Solve

$$y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2 \rightarrow (i)$$

$$\text{Solution: } y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2$$

$$E^2 y_n - 2E y_n + y_n = 2^n \cdot n^2$$

$$(E^2 - 2E + 1) y_n = 2^n \cdot n^2$$

Auxiliary equation is

$$E^2 - 2E + 1 = 0$$

$$(E - 1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

So C.F. is  $y_n = (C_1 + C_2 n)(1)^n$   
 $y_n = C_1 + C_2 n$

As for P.S. put  $y_n = 2^n (B_0 + B_1 n + B_2 n^2)$   
in (i) we have.

$$\begin{aligned} & 2^{(n+2)} (B_0 + B_1(n+2) + B_2(n+2)^2) - 2^{(n+1)} [2(B_0 + B_1(n+1) + B_2(n+1)^2)] \\ & + 2^n (B_0 + B_1 n + B_2 n^2) = 2^n \cdot n^2 \end{aligned}$$

$$\begin{aligned} & 2^n [4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2 \\ & - 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 \\ & + B_0 + B_1 n + B_2 n^2] = 2^n \cdot n^2 \end{aligned}$$

Dividing both sides by  $2^n$ 

$$\begin{aligned} & 4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0 \\ & - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0 + B_1 n + B_2 n^2 = n^2 \end{aligned}$$

Equating Coeff. of  $n^2$ :

$$4B_2 + B_2 - 4B_2 = 1$$

$$\Rightarrow B_2 = 1$$

Equating Coeff. of  $n$ :

$$4B_1 + 16B_2 - 4B_1 - 8B_2 + B_1 = 0$$

$$\Rightarrow 8B_2 + B_1 = 0$$

$$\Rightarrow 8(1) + B_1 = 0$$

$$\Rightarrow B_1 = -8$$



Type (4):When  $\phi(n) = a^n F(n)$ where  $F(n)$  is polynomial  
of degree  $t$ .

For P.S. trial substitution

$$y_n = a^n (B_0 + B_1 n + B_2 n^2 + \dots + B_t n^t)$$

If 'a' is root of auxiliary equation then  
above substitution fails.In this case multiply the trial  
function by suitable power of  $n$ .If 'a' is single root then multiply by  $n$ If 'a' is double root then multiply by  $n^2$ If 'a' is triple root then multiply by  $n^3$   
..... So on.Example: Solve

$$y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2 \rightarrow (i)$$

$$\text{Solution: } y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2$$

$$E^2 y_n - 2E y_n + y_n = 2^n \cdot n^2$$

$$(E^2 - 2E + 1) y_n = 2^n \cdot n^2$$

Auxiliary equation is

$$E^2 - 2E + 1 = 0$$

$$(E - 1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

So C.F. is  $y_n = (C_1 + C_2 n)(1)^n$   
 $y_n = C_1 + C_2 n$

As for P.S. put  $y_n = 2^n (B_0 + B_1 n + B_2 n^2)$   
in (i) we have.

$$\begin{aligned} & 2^{(n+2)} (B_0 + B_1(n+2) + B_2(n+2)^2) - 2 \left[ 2^{(n+1)} (B_0 + B_1(n+1) + B_2(n+1)^2) \right] \\ & + 2^n (B_0 + B_1 n + B_2 n^2) = 2^n \cdot n^2 \end{aligned}$$

$$2^n [4B_0 + 4nB_1 + 8B_2 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0 + B_1 n + B_2 n^2] = 2^n \cdot n^2$$

Dividing both sides by  $2^n$ 

$$\begin{aligned} & 4B_0 + 4nB_1 + 8B_2 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0 + B_1 n + B_2 n^2 = n^2 \end{aligned}$$

Equating Coeff. of  $n^2$ :

$$4B_2 + B_2 - 4B_2 = 1$$

$$\Rightarrow B_2 = 1$$

Equating coeff. of  $n$ :

$$4B_1 + 16B_2 - 4B_1 - 8B_2 + B_1 = 0$$

$$\Rightarrow 8B_2 + B_1 = 0$$

$$\Rightarrow 8(1) + B_1 = 0$$

$$\Rightarrow B_1 = -8$$



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Equating Constant term:

$$4B_0 + 16B_1 - 4B_0 - 4B_1 + B_0 - 4B_1 = 0$$

$$\Rightarrow B_0 + 12B_1 - 4B_1 = 0$$

$$\Rightarrow B_0 + 12(1) - 4(-8) = 0$$

$$\Rightarrow B_0 + 12 - 32 = 0$$

$$\Rightarrow B_0 - 20 = 0$$

$$\Rightarrow B_0 = 20$$

$$\text{P.S is } y_n = 2^n [B_0 + B_1 n + B_2 n^2] \\ = 2^n [20 - 8n + n^2]$$

General Sol = C.F + P.S

$$y_n = C_1 + C_2 n + 2^n (20 - 8n + n^2)$$

(Ans)

Example:-

$$\text{Solve } y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = 2^k \cdot k$$

Solution:-

$$y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = 2^k \cdot k \rightarrow (i)$$

$$E^3 y_k - 5E^2 y_k + 8E y_k - 4y_k = 2^k \cdot k$$

$$(E^3 - 5E^2 + 8E - 4)y_k = 2^k \cdot k$$

For C.F: Auxiliary equation is

$$E^3 - 5E^2 + 8E - 4 = 0$$

$$(E-1)(E^2 - 4E + 4) = 0$$

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$$\Rightarrow (E-1)(E-2)(E-2) = 0$$

$$\Rightarrow E = 1, 2, 2$$

So C.S is

$$y_k = C_1 (1)^k + (C_2 + C_3 k)(2)^k \rightarrow *$$

$$\text{Put } y_k = k^2 2^k (B_0 + B_1 k) \quad (\because k^2 \text{ because } 2 \text{ is double root})$$

Put in eq (i)

$$(k+3)^2 2^{k+3} (B_0 + B_1 (k+3)) - 5[(k+2)^2 2^{k+2} (B_0 + B_1 (k+2)) \\ + 8[(k+1)^2 2^{k+1} (B_0 + B_1 (k+1)) - 4k^2 2^k (B_0 + B_1 k)] = 2^k \cdot k$$

$$2^k [8(k+3)^2 (B_0 + B_1 k + 3B_1) - 5(4)(k+2)^2 (B_0 + B_1 k + 2B_1) \\ + 8(2)(k+1)^2 (B_0 + B_1 k + B_1) - 4k^2 (B_0 + B_1 k)] = 2^k \cdot k$$

Dividing by  $2^k$ 

$$8(k+3)^2 (B_0 + B_1 k + 3B_1) - 20(k+2)^2 (B_0 + B_1 k + 2B_1) \\ + 16(k+1)^2 (B_0 + B_1 k + B_1) - 4k^2 (B_0 + B_1 k) = k$$

Equating Coeff. of like power of  $k$ 

$$48B_0 + 216B_1 - 80B_0 - 240B_1 + 32B_1 + 48B_1 = 1 \rightarrow (2)$$

$$72B_0 + 216B_1 - 80B_0 - 160B_1 + 16B_1 + 16B_1 = 0 \rightarrow (3)$$

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$$② \Rightarrow 24 B_1 = 1$$

$$B_1 = \frac{1}{24}$$

$$③ \Rightarrow 8B_0 + 72B_1 = 0$$

$$8B_0 + 72\left(\frac{1}{24}\right) = 0$$

$$B_0 = -3/8$$

Particular Solution is

$$y_k = k^2 \cdot 2^k \left( \frac{-3}{8} + \frac{1}{24} k \right)$$

General solution of given difference equation is

$$y_k = C_1(1)^k + (C_2 + C_3 k) 2^k + k^2 \cdot 2^k \left( \frac{-3}{8} + \frac{1}{24} k \right)$$

(Ans)

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## EXERCISE

Solve

$$(i) \quad y_{n+2} - 9y_{n+1} + 20y_n = 3^n(n^2+1)$$

$$\text{Ans: } y_n = C_1 4^n + C_2 5^n + \left( \frac{37}{2} + \frac{9n}{2} + \frac{n^2}{2} \right) 3^n$$

$$(ii) \quad y_{k+2} - 2y_{k+1} + y_k = k2^k - 2^k$$

$$= (k-1)2^k = k2^k - 2^k$$

$$\text{Ans: } y_k = C_1 + C_2 k + (k-5)2^k$$

$$(iii) \quad U_{n+3} + 8U_n = (2n+3)2^n$$

$$\text{Ans: } U_n = C_1(-2)^n + 2^n \left( C_2 \cos \frac{\pi n}{3} + C_3 \sin \frac{\pi n}{3} \right) + \frac{1}{8} n 2^n$$

$$(iv) \quad y_{k+2} - 13y_{k+1} + 36y_k = 2^k(k^2+1)$$

$$\text{Ans: } C_1 4^k + C_2 9^k + 2^k \left( \frac{123}{343} + \frac{9k}{49} + \frac{1}{14} k^2 \right)$$

$$(v) \quad y_{k+2} + 6y_{k+1} + 25y_k = 2^k + k + 4$$

$$(vi) \quad (E^2 - 6E + 9)y_k = 3^k \cdot k^2$$

Ans:

$$y_k = (C_1 + C_2 k) 3^k + 3^k k^2 \left( \frac{1}{108} - \frac{1}{27} k + \frac{1}{108} k^2 \right)$$

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Type (5):

when  $\phi(n) = a \sin bn$  (or)  $\cos bn$   
also  $\phi(n) = \cos bn + \sin bn$

For P.S Trail substitution is

$$y_n = A \sin bn + B \cos bn$$

If trail substitution contain a term of C.F then multiply the trail substitution by suitable power of  $n$ .

Example:

$$y_{n+2} - 4y_{n+1} + 4y_n = \sin n \rightarrow (i)$$

Solution:

$$y_{n+2} - 4y_{n+1} + 4y_n = \sin n$$

$$E^2 y_n - 4E y_n + 4y_n = \sin n$$

$$(E^2 - 4E + 4)y_n = \sin n$$

Ch. eq. for C.S

$$E^2 - 4E + 4 = 0$$

$$(E-2)^2 = 0$$

$$\Rightarrow E = 2, 2$$

$$\text{C.F is } y_n = (C_1 + C_2 n)(2)^n \rightarrow (*)$$

For P.S Put  $y_n = A_1 \sin n + A_2 \cos n$  in

(i) we have

$$A_1 \sin(n+2) + A_2 \cos(n+2) - 4[A_1 \sin(n+1) + A_2 \cos(n+1)] + 4[A_1 \sin n + A_2 \cos n] = \sin n$$

$$A_1 [\sin(n) \cos(2) + \cos(n) \sin(2)] + A_2 [\cos(n) \cos(2) - \sin(n) \sin(2)] - 4A_1 [\sin(n) \cos(1) + \cos(n) \sin(1)] - 4A_2 [\cos(n) \cos(1) - \sin(n) \sin(1)] + 4A_1 \sin n + 4A_2 \cos n = \sin(n)$$

$$[A_1 \cos(2) - A_2 \sin(2) - 4A_1 \sin(1) - 4A_2 \sin(1) + 4A_1] \sin(n) + [A_1 \sin(2) + A_2 \cos(2) - 4A_1 \cos(1) - 4A_2 \cos(1) + 4A_2] \cos n = \sin(n)$$

Equating coeff of  $\sin(n)$ :

$$(\cos 2 - 4 \cos 1 + 4)A_1 - (\sin 2 + 4 \sin 1)A_2 = 1 \rightarrow (A)$$

Equating coeff of  $\cos(n)$ :

$$(\sin 2 - 4 \sin 1)A_1 + (\cos 2 - 4 \cos 1 + 4)A_2 = 0 \rightarrow (B)$$

$$\text{Let } K_1 = \cos 2 - 4 \cos 1 + 4$$

$$K_2 = \sin 2 + 4 \sin 1$$

then (A) and (B) becomes

$$K_1 A_1 - K_2 A_2 = 1 \Rightarrow K_1 A_1 - K_2 A_2 - 1 = 0$$

$$K_2 A_1 - K_1 A_2 = 0 \Rightarrow K_2 A_1 - K_1 A_2 - 0 = 0$$

$$\begin{matrix} A_1 & = & -A_2 & = & 1 \\ 0+K_1 & & 0+K_2 & & K_1^2 + K_2^2 \end{matrix}$$

$$\Rightarrow A_1 = \frac{K_1}{K_1^2 + K_2^2}, \quad A_2 = \frac{-K_2}{K_1^2 + K_2^2}$$



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P.S is  $y_n = A_1 \sin n + A_2 \cos n$   
 $= \frac{K_1}{k_1^2 + k_2^2} \sin n - \frac{K_2}{k_1^2 + k_2^2} \cos n \rightarrow *$

General Solution is (From \* and \*)

$$y_n = C.F + P.S$$

$$= (C_1 + C_2 n) 2^n + \frac{k_1}{k_1^2 + k_2^2} \sin n - \frac{k_2}{k_1^2 + k_2^2} \cos n$$

where  $k_1 = \cos 2 - 4 \cos 1 + 4$

$k_2 = \sin 2 - 4 \sin 1$

(Ans)

Example:

$$y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$$

Solution:  $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$

$$E^2 y_k - 2E y_k + y_k = \sin 5k + \cos 5k + 6$$

For  $(E^2 - 2E + 1) y_k = \sin 5k + \cos 5k + 6$

C.S Auxiliary equation is

$$E^2 - 2E + 1 = 0$$

$$(E - 1)^2 = 0$$

$$E = 1, 1$$

$$y_k = (C_1 + C_2 k) (1)^k \rightarrow (*)$$

For P.S put  $y_k^* = A \sin 5k + B \cos 5k + C k^2$

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Put in  $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$   
 we have.

$$A \sin 5(k+2) + B \cos 5(k+2) + C(k+2)^2$$

$$- 2[A \sin 5(k+1) + B \cos 5(k+1) + C(k+1)^2] +$$

$$A \sin 5k + B \cos 5k + Ck^2 = \sin 5k + \cos 5k + 6$$

$$A[\sin 5k \cos 10 + \cos 5k \sin 10] +$$

$$B[\cos 5k \cos 10 - \sin 5k \sin 10] + C[k^2 + 4k + 4]$$

$$- 2A[\sin 5k \cos 5 + \cos 5k \sin 5] -$$

$$2B[\cos 5k \cos 5 - \sin 5k \sin 5] - 2C[k^2 + 2k + 1]$$

$$+ A \sin 5k + B \cos 5k + Ck^2 = \sin 5k + \cos 5k + 6$$

Comparing co-efficients of  $\sin 5k$ ,  $\cos 5k$  and constant, we have,

$$(\cos 10 - 2 \cos 5 + 1)A - (\sin 10 - 2 \sin 5)B = 1 \rightarrow (i)$$

$$(\sin 10 - 2 \sin 5)A + (\cos 10 - 2 \cos 5 + 1)B = 1 \rightarrow (ii)$$

and

$$C(k^2 + 4k + 4 - 2k^2 - 4k - 2 + k^2) = 6$$

$$C(2) = 6$$

$$\Rightarrow C = 3$$

Now (i) and (ii) written as

$$k_1 A - k_2 B = 1 \Rightarrow k_1 A - k_2 B - 1 = 0$$

$$k_2 A + k_1 B = 1 \Rightarrow k_2 A + k_1 B - 1 = 0$$

Where  $k_1 = \cos 10 - 2 \cos 5 + 1$

$k_2 = \sin 10 - 2 \sin 5$

Solving these eq. by cross-multiplication.

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$$\frac{A}{K_2 + K_1} = \frac{-B}{-K_1 + K_2} = \frac{1}{K_1^2 + K_2^2}$$

$$\Rightarrow A = \frac{K_1 + K_2}{K_1^2 + K_2^2}, \quad B = \frac{K_1 - K_2}{K_1^2 + K_2^2}$$

So P.S is

$$y_k = \left( \frac{K_1 + K_2}{K_1^2 + K_2^2} \right) \sin 5k + \left( \frac{K_1 - K_2}{K_1^2 + K_2^2} \right) \cos 5k + 3k^2 \rightarrow (*)$$

The general solution by (\*) and (\*)'

$$y_k = (C_1 + C_2 k) \left( \frac{K_1 + K_2}{K_1^2 + K_2^2} \right) \sin 5k + \left( \frac{K_1 - K_2}{K_1^2 + K_2^2} \right) \cos 5k + 3k^2$$

$$\text{Where } K_1 = \cos 10 - 2\cos 5 + 1$$

$$K_2 = \sin 10 - 2\sin 5$$

(Ans)

## EXERCISE

Solve the following difference Equation.

1)  $y_{n+2} - 3y_{n+1} - 4y_n = \sin 2n$

Ans:  $y_n = C_1 4^n + C_2 (-1)^n + \frac{K_1 \sin 2n - K_2 \cos 2n}{K_1^2 + K_2^2}$

where  $K_1 = \cos 4 - 3\cos 2 - 4$   
 $K_2 = \sin 4 - 3\sin 2$

2)  $y_{n+2} - 2y_{n+1} + y_n = \sin 5n + \cos 5n + 9$

3)  $y_{n+2} + y_n = \sin n\pi$   
 Ans:  $\frac{C_1 \cos n\pi}{2} + \frac{C_2 \sin n\pi}{2} - \frac{1}{2} \frac{n \sin n\pi}{2}$

4)  $y_{n+2} - 8y_{n+1} + y_n = 2^n + \sin n$

5)  $y_{n+4} - 6y_n = \sin 3n$

6)  $y_{n+2} + y_n = \sin x$

7)  $y_{n+2} + y_n = \frac{\cos n}{2}$

8)  $y_{k+2} - 7y_{k+1} + 12y_k = \cos k$



## Type (6):

When  $\phi(n) = a^n (\cos An \text{ or } \sin An)$

where  $A$  is constant.

In order to find particular solution we shall make substitution

$y_n = a^n (C_1 \sin An + C_2 \cos An)$   
and find value of  $C_1$  and  $C_2$ .

Example:

$$y_{t+2} - 7y_{t+1} - 8y_t = 7^t (\cos 3t + \sin 3t) \rightarrow (i)$$

Solution:-  $y_{t+2} - 7y_{t+1} - 8y_t = 7^t (\cos 3t + \sin 3t)$

$$\Rightarrow E^2 y_t - 7E y_t - 8y_t = 7^t (\cos 3t + \sin 3t)$$

$$\Rightarrow (E^2 - 7E - 8) y_t = 7^t (\cos 3t + \sin 3t)$$

Auxiliary equation is

$$E^2 - 7E - 8 = 0$$

$$\Rightarrow E^2 - 8E + E - 8 = 0$$

$$\Rightarrow E(E-8) + 1(E-8) = 0$$

$$\Rightarrow (E-8)(E+1) = 0$$

$$\Rightarrow E = -1, 8$$

C.F is  $y_t = C_1(-1)^t + C_2(8)^t \rightarrow (*)$

For P.S

put  $y_t = 7^t [A_1 \cos 3t + A_2 \sin 3t]$

put in eq. (i)

$$\begin{aligned} & 7^t [A_1 \cos 3(t+2) + A_2 \sin 3(t+2)] \\ & - 7 \cdot 7^{t+1} [A_1 \cos 3(t+1) + A_2 \sin 3(t+1)] \\ & - 8 [7^t (A_1 \cos 3t + A_2 \sin 3t)] = 7^t (\cos 3t + \sin 3t) \end{aligned}$$

Dividing both sides by  $7^t$

$$\begin{aligned} & 49A_1 [\cos 3t \cos 6 - \sin 3t \sin 6] + \\ & 49A_2 [\sin 3t \cos 6 + \cos 3t \sin 6] - \\ & 49A_1 [\cos 3t \cos 3 - \sin 3t \sin 3] - \\ & 49A_2 [\sin 3t \cos 3 - \cos 3t \sin 3] \\ & - 8A_1 \cos 3t - 8A_2 \sin 3t = \cos 3t + \sin 3t \end{aligned}$$

Comparing coeffs of  $\cos 3t$  and  $\sin 3t$

$$\begin{aligned} (49 \cos 6 - 49 \cos 3 - 8)A_1 + (49 \sin 6 - 49 \sin 3)A_2 &= 1 \\ -(49 \sin 6 - 49 \sin 3)A_1 + (49 \cos 6 - 49 \cos 3 - 8)A_2 &= 1 \end{aligned}$$

$$\text{Let } K_1 = 49 \cos 6 - 49 \cos 3 - 8$$

$$K_2 = 49 \sin 6 - 49 \sin 3$$

$$\Rightarrow K_1 A_1 + K_2 A_2 - 1 = 0$$

$$-K_2 A_1 + K_1 A_2 - 1 = 0$$

$$\begin{aligned} A_1 &= \frac{-A_2}{-K_2 + K_1} = \frac{1}{K_1^2 + K_2^2} \end{aligned}$$

$$\Rightarrow A_1 = \frac{K_1 - K_2}{K_1^2 + K_2^2}, A_2 = \frac{K_1 + K_2}{K_1^2 + K_2^2}$$

So P.S is



$$y_t = 7^t \left[ \left( \frac{k_1 - k_2}{k_1^2 + k_2^2} \right) \cos 3t + \left( \frac{k_1 + k_2}{k_1^2 + k_2^2} \right) \sin 3t \right] \rightarrow (x')$$

By \* and x' the general solution is

$$y_t = C.F + P.S \\ = C_2 8^t + C_1 (-1)^t + 7^t \left[ \frac{(k_1 - k_2) \cos 3t + (k_1 + k_2) \sin 3t}{k_1^2 + k_2^2} \right]$$

Example:-

(Ans)

Solve  $y_{k+2} + 13y_{k+1} + 3y_k = 3^k \cos 4k \rightarrow (i)$

Solution:-

$$\text{Given } y_{k+2} + 13y_{k+1} + 3y_k = 3^k \cos 4k$$

$$E^2 y_k + 13E y_k + 3y_k = 3^k \cos 4k$$

$$(E^2 + 13E + 3)y_k = 3^k \cos 4k$$

Auxiliary equation

$$E^2 + 13E + 3 = 0$$

$$E = \frac{-13 \pm \sqrt{9 - 4(13)(1)}}{2}$$

$$= \frac{-13 \pm \sqrt{157}}{2}$$

So complementary solution is

$$y_k = A_1 \left( \frac{-13 + \sqrt{157}}{2} \right)^k + A_2 \left( \frac{-13 - \sqrt{157}}{2} \right)^k \rightarrow (*)$$

For P.S put  $y_k = 3^k [C_1 \sin 4k + C_2 \cos 4k]$   
put in (i) we have

$$\begin{aligned} & 3^{k+2} [C_1 \sin 4(k+2) + C_2 \cos 4(k+2)] + \\ & 13 \cdot 3^{k+1} [C_1 \sin 4(k+1) + C_2 \cos 4(k+1)] + \\ & 3 \cdot 3^k [C_1 \sin 4k + C_2 \cos 4k] = 3^k \cos 4k \end{aligned}$$

Dividing both sides by  $3^k$  we get

$$\begin{aligned} & (3)C_1 [\sin 4k \cos 8 + \cos 4k \sin 8] + \\ & 9C_2 [\cos 4k \cos 8 - \sin 4k \sin 8] + \\ & 39C_1 [\sin 4k \cos 4 + \cos 4k \sin 4] + \\ & 39C_2 [\cos 4k \cos 4 - \sin 4k \sin 4] + \\ & 3C_1 \sin 4k + 3C_2 \cos 4k = \cos 4k \end{aligned}$$

Comparing Co-efficient of  $\cos 4k$  and  $\sin 4k$

$$C_1 [9 \cos 8 + 39 \cos 4 + 3] - C_2 [9 \sin 8 + 39 \sin 4] = 0$$

$$C_1 [9 \sin 8 + 39 \sin 4] + C_2 [9 \cos 8 + 39 \cos 4 + 3] = 1$$

$$\text{Let } 9 \cos 8 + 39 \cos 4 + 3 = k_1$$

$$9 \sin 8 + 39 \sin 4 = k_2$$

then

$$k_1 C_1 - k_2 C_2 - 0 = 0$$

$$k_2 C_1 + k_1 C_2 - 1 = 0$$

$$\frac{C_1}{k_2} = \frac{-C_2}{-k_1} = \frac{1}{k_1^2 + k_2^2}$$

$$\text{So } C_1 = \frac{k_2}{k_1^2 + k_2^2}, \quad C_2 = \frac{k_1}{k_1^2 + k_2^2}$$

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So P.S is

$$y_k = 3^k \left[ \frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right] \rightarrow (*)'$$

by \* and \*' general solution is

$$y_k = A_1 \left( \frac{-13 + \sqrt{157}}{2} \right) + A_2 \left( \frac{-13 - \sqrt{157}}{2} \right) + 3^k \left[ \frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right]$$

(Ans.)

Example:

$$\text{Solution: } y_{k+2} + y_{k+1} + y_k = 2^k \cdot \sin k \cos 3k$$

$$\text{Given } y_{k+2} + y_{k+1} + y_k = 2^k \cdot \sin k \cos 3k$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [2 \sin k \cos 3k]$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [\sin 2k - \sin k] \rightarrow (1)$$

For C.F

$$y_{k+2} + y_{k+1} + y_k = 0$$

$$E^2 y_k + E y_k + y_k = 0$$

$$(E^2 + E + 1) y_k = 0 \quad \text{As } y_k \neq 0$$

$$\Rightarrow E^2 + E + 1 = 0$$

$$E = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

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So Complementary Sol is

$$y_k = R^k [A_1 \cos k\theta + A_2 \sin k\theta]$$

$$\text{where } R = \sqrt{\frac{1+3}{4}} = 1$$

$$\theta = \tan^{-1} \frac{\sqrt{3}}{-1} = 120^\circ$$

$$y_k = (1)^k [A_1 \cos 120k + A_2 \sin 120k] \rightarrow (*)$$

For particular solution we divide (1) into two equations.

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} \sin 2k \rightarrow (A)$$

$$\text{and } y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} \sin k \rightarrow (B)$$

First we solve (A)

$$\text{put } y_k = 2^k (C_1 \sin 2k + C_2 \cos 2k)$$

$$(A) \Rightarrow$$

$$\frac{2^{k+2}}{2} (C_1 \sin 2(k+2) + C_2 \cos 2(k+2)) +$$

$$\frac{2^{k+1}}{2} (C_1 \sin 2(k+1) + C_2 \cos 2(k+1)) +$$

$$\frac{2^k}{2} (C_1 \sin 2k + C_2 \cos 2k) = \frac{2^k}{2} \sin 2k$$

Dividing by  $2^k$  and comparing coeff. of  $\sin 2k$  and  $\cos 2k$  we have

$$(4 \cos 4 + 2 \cos 2 + 1) C_1 - (4 \sin 4 + 2 \sin 2) C_2 = \frac{1}{2}$$

$$(4 \sin 4 + 2 \sin 2) C_1 + (4 \cos 4 + 2 \cos 2 + 1) C_2 = 0$$



So P.S is

$$y_k = 3^k \left[ \frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right] \rightarrow (*)'$$

by \* and \*' general solution is

$$y_k = A_1 \left( \frac{-13 + \sqrt{157}}{2} \right) + A_2 \left( \frac{-13 - \sqrt{157}}{2} \right) + 3^k \left[ \frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right]$$

(Ans).

Example:

$$y_{k+2} + y_{k+1} + y_k = 2^k \cdot \sin k \cos 3k$$

Solution:

$$\text{Given } y_{k+2} + y_{k+1} + y_k = 2^k \cdot \sin k \cos 3k$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [2 \sin k \cos 3k]$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [\sin 2k - \sin k] \rightarrow (1)$$

For C.F

$$y_{k+2} + y_{k+1} + y_k = 0$$

$$E^2 y_k + E y_k + y_k = 0$$

$$(E^2 + E + 1) y_k = 0 \quad \text{As } y_k \neq 0$$

$$\Rightarrow E^2 + E + 1 = 0$$

$$E = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

So Complementary Sol is

$$y_k = R^k [A_1 \cos k\theta + A_2 \sin k\theta]$$

$$\text{where } R = \sqrt{\frac{1+3}{4}} = 1$$

$$\theta = \tan^{-1} \frac{\sqrt{3}}{-1} = 120^\circ$$

$$y_k = (1)^k [A_1 \cos 120k + A_2 \sin 120k] \rightarrow (*)$$

For particular solution we divide (1) into two equations.

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} \sin 2k \rightarrow (A)$$

$$\text{and } y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} \sin k \rightarrow (B)$$

First we solve (A)

$$\text{put } y_k = 2^k (C_1 \sin 2k + C_2 \cos 2k)$$

(A)  $\Rightarrow$ 

$$\begin{aligned} & 2^{k+2} (C_1 \sin 2(k+2) + C_2 \cos(k+2)) + \\ & \cdot 2^{k+1} (C_1 \sin 2(k+1) + C_2 \cos(k+1)) + \\ & 2^k (C_1 \sin 2k + C_2 \cos 2k) = \frac{2^k}{2} \sin 2k \end{aligned}$$

Dividing by  $2^k$  and comparing coeff. of  $\sin 2k$  and  $\cos 2k$  we have

$$\begin{aligned} (4 \cos 4 + 2 \cos 2 + 1) C_1 - (4 \sin 4 + 2 \sin 2) C_2 &= \frac{1}{2} \\ (4 \sin 4 + 2 \sin 2) C_1 + (4 \cos 4 + 2 \cos 2 + 1) C_2 &= 0 \end{aligned}$$



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$$\text{Let } k_1 = 4\cos 4 + 2\cos 2 + 1$$

$$k_2 = 4\sin 4 + 2\sin 2$$

then above write as

$$k_1 C_1 - k_2 C_2 = 1/2$$

$$k_2 C_1 + k_1 C_2 = 0$$

$$\Rightarrow k_1 C_1 - k_2 C_2 - 1/2 = 0$$

$$k_2 C_1 + k_1 C_2 - 0 = 0$$

by solving

$$\begin{matrix} C_1 & = & -C_2 & = & 1 \\ -k_1/2 & & -k_2/2 & & k_1^2 + k_2^2 \end{matrix}$$

$$\Rightarrow C_1 = \frac{k_1}{2(k_1^2 + k_2^2)}, \quad C_2 = \frac{-k_2}{2(k_1^2 + k_2^2)}$$

So particular solution is

$$y_k = 2^k \left[ \frac{k_1 \sin 2k - k_2 \cos 2k}{2(k_1^2 + k_2^2)} \right]$$

Similarly For (B) P.S is

$$y_k = 2^k \left[ \frac{k_1 \sin k - k_2 \cos k}{2(k_1^2 + k_2^2)} \right]$$

Particular solution of (A) and (B) is actually

P.S of (i)

So P.S of (i) = P.S of (A) - P.S of (B)

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$$y_k = 2^k \left[ \frac{k_1 (\sin 2k - \sin k) - k_2 (\cos 2k - \cos k)}{2(k_1^2 + k_2^2)} \right] \rightarrow (*)'$$

General solution of eq (i) is (by \* and \*')

$$y_k = (1)^k [A_1 \cos 120^\circ k + A_2 \sin 120^\circ k] +$$

$$2^k \left[ \frac{k_1 (\sin 2k - \sin k) - k_2 (\cos 2k - \cos k)}{2(k_1^2 + k_2^2)} \right]$$

(Answer)

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# SIMULTANEOUS LINEAR

## DIFFERENCE EQUATIONS

If two or more difference equations are given with same number of unknown functions, then we can solve such equations simultaneously by using a procedure which eliminates all but one of the unknown.

Example: Solve the system

$$\begin{aligned} U_{n+1} - U_n + 3V_n &= 7 \\ 3V_{n+1} + V_n - 2U_n &= 6 \end{aligned}$$

Solution:

Given  $U_{n+1} - U_n + 3V_n = 7$

$$3V_{n+1} + V_n - 2U_n = 6$$

In operator notation, the above equations can be written as

$$E U_n - U_n + 3V_n = 7$$

$$\Rightarrow (E-1)U_n + 3V_n = 7 \rightarrow (i)$$

and  $3V_{n+1} + V_n - 2U_n = 6$

$$\dots 3EV_n + V_n - 2U_n = 6$$

$$(3E+1)V_n - 2U_n = 6 \rightarrow (ii)$$

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Multiplying eq (i) by  $(3E+1)$  and eq (ii) by 3 then subtract them. we get

$$\begin{aligned} (3E+1)(E-1)U_n + 3(3E+1)V_n &= 7(3E+1) \\ - 6U_n + 3(3E+1)V_n &= +18 \\ + & \quad \quad \quad - \end{aligned}$$

$$(3E^2 - 2E - 1)U_n + 6U_n = 21E + 7 - 18$$

$$(3E^2 - 2E + 5)U_n = 21 + 7 - 18$$

$$(3E^2 - 2E + 5)U_n = 10$$

$$\Rightarrow 3U_{n+2} - 2U_{n+1} + 5U_n = 10 \rightarrow (3)$$

which is a difference equation (Non-homogeneous) to solve this we find C.S and P.S of (3)

For complementary Solution

$$(3E^2 - 2E + 5)U_n = 0$$

$$\Rightarrow 3E^2 - 2E + 5 = 0$$

$$\Rightarrow E = \frac{2 \pm \sqrt{4 - 4(3)(5)}}{6}$$

$$E = \frac{2 \pm \sqrt{56}}{6} = \frac{1 \pm \sqrt{14}}{3}$$

So C.S is

$$U_n = R^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

$$R = \sqrt{\frac{1}{9} + \frac{14}{9}} = \sqrt{\frac{15}{9}} = \frac{\sqrt{15}}{3}$$

$$\theta = \tan^{-1}(\sqrt{14})$$

$$\Rightarrow U_n = \left(\frac{\sqrt{15}}{3}\right)^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

$$\therefore \theta = \tan^{-1}(\sqrt{14})$$

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To find particular solution of (3) Put

$$U_n = C$$

$$(3) \Rightarrow 3C - 2C + 5C = 10$$

$$6C = 10$$

$$C = 5/3$$

$$U_n^* = 5/3$$

$$\text{General Solution} = U_n + U_n^* \\ = \left(\frac{\sqrt{15}}{3}\right)^n [A_1 \cos n\theta + A_2 \sin n\theta] + 5/3$$

$$\text{where } \theta = \tan^{-1}(\sqrt{14})$$

Which is solution of  $U_n$ . (Ans.)

To find  $V_n$ , Put value of  $U_n$  in (i)

$$3V_n = 7 - (E-1)U_n \\ = 7 - U_{n+1} + U_n$$

$$V_n = \frac{7}{3} - \frac{1}{3} \left(\frac{\sqrt{15}}{3}\right)^{n+1} [A_1 \cos(n+1)\theta + A_2 \sin(n+1)\theta] \\ + \left(\frac{\sqrt{15}}{3}\right)^n [A_1 \cos n\theta + A_2 \sin n\theta] + \frac{5}{3}$$

Which is required solution for  $V_n$ .

(Ans.)

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Example:

Solve the system of Equation

$$U_{n+1} + V_n - 3U_n = n$$

$$3U_n + V_{n+1} - 5V_n = 4^n$$

Solution:- Given

$$U_{n+1} + V_n - 3U_n = n \rightarrow (i)$$

$$V_{n+1} - 5V_n + 3U_n = 4^n \rightarrow (ii)$$

In operator notation, above eq. (i) & (ii) becomes

$$EU_n + V_n - 3U_n = n$$

$$EV_n - 5V_n + 3U_n = 4^n$$

$$\text{and } (E-3)U_n + V_n = n \rightarrow (iii)$$

$$(E-5)V_n + 3U_n = 4^n \rightarrow (iv)$$

Multiplying (iii) by  $(E-5)$  then subtract from (iv)

$$(E-5)(E-3)U_n + (E-5)V_n = (E-5)n \\ + 3U_n + (E-5)V_n = +4^n$$

$$(E^2 - 8E + 15)U_n - 3U_n = -5n + En - 4^n$$

$$(E^2 - 8E + 12)U_n = n+1 - 5n - 4^n$$

$$(\because En = n+1)$$

$$(E^2 - 8E + 12)U_n = 1 - 4n - 4^n \rightarrow (v)$$

Auxiliary Equation is  $E^2 - 8E + 12 = 0$

$$E^2 - 2E - 6E + 12 = 0$$

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$$E(E-2) - 6(E-2) = 0$$

$$(E-2)(E-6) = 0$$



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$$\Rightarrow E = 2, 6$$

So C.F. is

$$U_n = C_1 (2)^n + C_2 (6)^n \longrightarrow (*)$$

Eq. (v) be write it as

$$U_{n+2} - 8U_{n+1} + 12U_n = 1 - 4n - 4^n$$

For P.S consider two equations

$$U_{n+2} - 8U_{n+1} + 12U_n = 1 - 4n \longrightarrow \textcircled{A}$$

$$U_{n+2} - 8U_{n+1} + 12U_n = -4^n \longrightarrow \textcircled{B}$$

Now consider  $\textcircled{A}$  For P.S

$$\text{Put } U_n = B_0 + B_1 n$$

Put in  $\textcircled{A}$ 

$$\Rightarrow B_0 + B_1(n+2) - 8B_0 + 8B_1(n+1) + 12B_0 + 12B_1 n = 1 - 4n$$

$$\Rightarrow B_0 + B_1 n + 2B_1 - 8B_0 - 8B_1 n - 8B_1 + 12B_0 + 12B_1 n = 1 - 4n$$

Equating coeff of  $n$  & constant term.

$$B_1 - 8B_1 + 12B_1 = -4$$

$$5B_1 = -4$$

$$B_1 = -4/5$$

and

$$B_0 + 2B_1 - 8B_0 - 8B_1 + 12B_0 = 1$$

$$5B_0 - 6B_1 = 1$$

$$5B_0 - 6\left(-\frac{4}{5}\right) = 1$$

$$5B_0 = 1 - \frac{24}{5}$$

$$B_0 = \frac{-19}{25}$$

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So P.S for  $\textcircled{A}$  is

$$U_n = \frac{-19 - 4n}{25 \cdot 5}$$

For eq.  $\textcircled{B}$  Let  $U_n = C 4^n$ 

$$U_{n+1} = C 4^{n+1}$$

$$U_{n+2} = C 4^{n+2}$$

put in  $\textcircled{B}$ 

$$C 4^{n+2} - 8C 4^{n+1} + 12C 4^n = -4^n$$

$$4^n (16C - 32C + 12C) = 4^n (-1)$$

$$16C - 32C + 12C = -1$$

$$-4C = -1$$

$$C = 1/4$$

$$\text{P.S is } U_n = \frac{1}{4} 4^n$$

So P.S for (v) is = P.S of  $\textcircled{A}$  + P.S of  $\textcircled{B}$ 

$$U_n = \frac{-19 - 4n}{25 \cdot 5} + \frac{1}{4} 4^n \longrightarrow (*)'$$

General solution of  $U_n$  is (by \* & \*')

$$U_n = C_1 (2)^n + C_2 (6)^n - \frac{19 - 4n}{25 \cdot 5} + \frac{1}{4} 4^n$$

which is required solution for  $U_n$ .

Now; by eq. (iii),

$$V_n = n - U_{n+1} - 3U_n$$

$$= n - \left[ C_1 (2)^{n+1} + C_2 (6)^{n+1} - \frac{19 - 4(n+1)}{25 \cdot 5} + \frac{1}{4} 4^{(n+1)} \right] - 3 \left[ C_1 (2)^n + C_2 (6)^n - \frac{19 - 4n}{25 \cdot 5} + \frac{1}{4} 4^n \right]$$

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$$V_n = n + (-2C_1 + 3C_2)2^n + (-6C_2 + 3C_1)6^n \\ + \left(\frac{19}{25} - \frac{57}{25} + \frac{4}{5}\right) + \left(\frac{4}{5} - \frac{12}{5}\right)n \\ + \frac{(-1+3)}{4}4^n$$

$$V_n = n + C_1 2^n - 3C_2 6^n - \frac{18}{25} - \frac{8n}{5} - \frac{4}{4}$$

$$V_n = C_1 2^n - 3C_2 6^n - 3n - \frac{18}{25} - \frac{4}{4}$$

which is required solution for  $V_n$ .

(Answer)

Example: Solve system of equations

$$U_{n+1} - U_n + V_n = 7$$

$$3V_{n+1} - 2V_n + U_n = 2$$

Solution:- Given  $U_{n+1} - U_n + V_n = 7$

$$3V_{n+1} - 2V_n + U_n = 2$$

In operator notation, we can write

$$EU_n - U_n + V_n = 7$$

$$\Rightarrow (E-1)U_n + V_n = 7 \rightarrow (i)$$

$$\Rightarrow (3E-2)V_n + U_n = 2 \rightarrow (ii)$$

Multiplying (ii) by  $(E-1)$  and subtract from (i)

$$(3E-2)(E-1)V_n + (E-1)U_n = 2(E-1) \\ + V_n + (E-1)U_n = 7$$

$$(3E^2 - 5E + 2)V_n = 2 - 2 - 7$$

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$$\Rightarrow (3E^2 - 5E + 1)V_n = -7 \\ 3V_{n+2} - 5V_{n+1} + V_n = -7 \rightarrow (iii)$$

Auxiliary equation is  $3E^2 - 5E + 1 = 0$   
 $\Rightarrow E = \frac{5 \pm \sqrt{13}}{6}$

then C.F  $V_n = C_1 \left(\frac{5+\sqrt{13}}{6}\right)^n + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^n$

To find P.S Put  $V_n = C$  in (iii)

$$3C - 5C + C = -7$$

$$-C = -7$$

$$C = 7$$

P.S of  $V_n = 7$

General Solution = C.F + P.S

$$= C_1 \left(\frac{5+\sqrt{13}}{6}\right)^n + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^n + 7$$

which is required solution for  $V_n$ .

Now

$$U_n = 2 - 3V_{n+1} + 2V_n$$

$$= 2 - 3 \left[ C_1 \left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^{n+1} + 7 \right]$$

$$+ 2 \left[ C_1 \left(\frac{5+\sqrt{13}}{6}\right)^n + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^n + 7 \right]$$

$$U_n = 2 - 3 \left[ C_1 \left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^{n+1} \right]$$

$$+ 2 \left[ C_1 \left(\frac{5+\sqrt{13}}{6}\right)^n + C_2 \left(\frac{5-\sqrt{13}}{6}\right)^n \right] - 7$$

which is required solution for  $U_n$ .

(Answer)

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## EXERCISES

Solve the following Simultaneous difference equation.

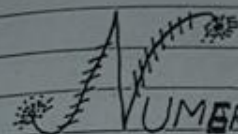
$$(i) \quad \begin{aligned} 3U_n + 4V_{n+1} - 5V_n &= 2^n \\ 2U_{n+1} + 3U_n + 4V_n &= 7 \end{aligned}$$

$$(ii) \quad \begin{aligned} U_{n+1} - U_n + V_n &= 2^n \\ 3V_{n+1} + 2V_n + U_n &= 7 \end{aligned}$$

$$(iii) \quad \begin{aligned} 3V_{n+1} - 2V_n + U_n &= n^2 \\ U_{n+1} - 3U_n + V_n &= 3^n \end{aligned}$$

$$(iv) \quad \begin{aligned} U_{n+1} - U_n + V_n &= 1 \\ 3V_{n+1} - 2V_n + U_n &= 2 \end{aligned}$$

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## NUMERICAL METHODS

## FOR ORDINARY DIFFERENTIAL EQUATION

An Ordinary differential equation is a relation between a function, its derivatives and variable upon which they depend.

The most general form of an O.D.E is  $\phi(y, y', y'', \dots, y^{(n)}) = 0$ .

where  $y$  and its derivatives are all function of  $x$ .

In this chapter we shall derive and analyse numerical methods for solving problem of ordinary diff. equation.

The main form of problem that we shall study is initial value problem,

$$y' = f(x, y) \quad ; \quad y(x_0) = y_0$$

The important methods of solving O.D.E's of first order are as follows.

(i) Euler's Method.

(ii) Heun's Method (Improved Euler's Method).

(iii) Modified Euler's Method.

(iv) Taylor's Series Method.

(v) Runge-Kutta Method.

(vi) Adams-Bashforth method.

etc.



# 1. EULER'S METHOD :-

Consider the first order differential equation together with an initial conditions is

$$\frac{dy}{dx} = f(x, y) \rightarrow (i)$$

$$; y(x_0) = y_0$$

By Taylor Series

$$y(x_0 + h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots$$

$$y(x_0 + h) = y(x_0) + h y'(x_0)$$

∴ Neglecting  $h^2$  and higher power of  $h$ .

$$y(x_1) = y(x_0) + h f(x_0, y_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly,  $y_2 = y_1 + h f(x_1, y_1)$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

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It is Euler's Iterative formula.

Remark:

Drawback of Euler's Method, if  $h$  is small ( $h$  is unit stepsize), Euler's Method is too slow. However if  $h$  is large this method is inaccurate.

Example :-

Solve  $\frac{dy}{dx} = x + y$  where  $y(0) = 1$   
;  $h = 0.2$

Find  $y(0.4)$  by Euler's Method and Compare with exact value.

Solution :-

The given initial value problem is

$$\frac{dy}{dx} = x + y \rightarrow (i)$$

$$; y(0) = 1, h = 0.2$$

The Euler's Formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\text{As } f(x, y) = x + y$$

$$f(x_n, y_n) = x_n + y_n$$

So

$$y_{n+1} = y_n + h f(x_n, y_n) \rightarrow (ii)$$

For  $n=0$ , we have From (ii)

$$y_1 = y_0 + h f(x_0, y_0), \quad y_1 = y(x_1)$$

$$= 1 + 0.2(0 + 1) \quad x_1 = x_0 + h$$

$$= 1 + 0.2 \quad = 0 + 0.2$$

$$= 1.2 \quad x_1 = 0.2$$

$$y(0.2) = 1.2$$

For  $n=1$ :

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 1.2 + 0.2(0.2 + 1.2)$$

$$= 1.2 + 0.2(1.4) \quad x_2 = x_0 + 2h$$

$$= 1.2 + 0.28 \quad x_2 = 0 + 2(0.2)$$

$$= 1.48 \quad x_2 = 0.4$$

$$y(0.4) = 1.48$$

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Exact Solution:

Given O.D.E is

$$\frac{dy}{dx} = x + y \rightarrow (3)$$

$$\frac{dy}{dx} - y = x \rightarrow (3') \text{ (Linear Diff. Eq.)}$$

$$\text{I.F is } e^{\int -1 dx} = e^{-x}$$

Then (3)' becomes

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} x$$

$$\frac{d}{dx} (y e^{-x}) = x e^{-x}$$

Integrating both sides

$$y e^{-x} = -x e^{-x} - e^{-x} + C$$

$$\Rightarrow y e^{-x} = -e^{-x}(x+1) + C$$

$$\Rightarrow y = -(x+1) + C e^x$$

$$\Rightarrow y = C e^x - (x+1) \rightarrow (4)$$

Apply condition  $y(0) = 1$ 

$$1 = C - 1$$

$$\Rightarrow C = 2$$

then (4) becomes

$$y = 2e^x - (x+1)$$

at  $x = 0.4$ 

$$y = 2e^{(0.4)} - (0.4+1)$$

$$y = 1.5836$$

So error = Exact - Apprx. value

$$= 1.5836 - 1.48$$

$$E = 0.1036$$

(Answer)

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Example:

Find solution of  $\frac{dy}{dx} = +2xy^2$ ,  
 $y(0) = 1$  &  $h = 0.05$ Find  $y(0.3)$  by using Euler's Method.Solution:- The given initial <sup>value</sup> problem is  
 $\frac{dy}{dx} = +2xy^2 \rightarrow (1)$ 

$$y(0) = 1 \text{ i.e. } x_0 = 0, y_0 = 1$$

$$h = 0.05$$

Euler's Formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where  $f(x, y) = +2xy^2$ 

$$f(x_n, y_n) = 2x_n y_n^2$$

then

$$y_{n+1} = y_n + h(2x_n y_n^2) \rightarrow (2)$$

Put  $n=0$  in (2):

$$y_1 = y_0 + h(2x_0 y_0^2)$$

$$= 1 + 0.05(2(0)(1)^2)$$

$$= 1 + 0.05(0)$$

$$y(x_1) = 1$$

$$y(x_1) = y_1$$

$$x_1 = x_0 + h$$

$$= 0 + 0.05 = 0.05$$

$$\Rightarrow y(0.05) = 1$$

Put  $n=1$  in (2):

$$y_2 = y_1 + h(2x_1 y_1^2)$$

$$= 1 + 0.05(2(0.05)(1)^2)$$

$$= 1 + 0.05$$

$$y_2 = 1.0050$$

and  $x_2 = x_0 + 2h$ 

$$= 0 + 2(0.05) = 0.100$$

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$$\Rightarrow y(x_1) = y_1$$

$$\Rightarrow y(0.100) = 1.0050$$

Put  $n=2$  in (2):

$$y_2 = y_1 + h(2x_1 y_1^2)$$

$$= 1.0050 + 0.05(2(0.100)(1.0050)^2)$$

$$= 1.0151$$

$$\text{and } x_2 = x_0 + 2h$$

$$= 0 + 2(0.05)$$

$$x_2 = 0.1000$$

$$\Rightarrow y(0.1500) = 1.0151$$

Put  $n=3$  in (2):

$$y_3 = y_2 + h(2x_2 y_2^2)$$

$$= 1.0151 + 0.05(2(0.1500)(1.0151)^2)$$

$$= 1.0306$$

$$\text{and } x_3 = x_0 + 3h$$

$$= 0 + 3(0.05)$$

$$x_3 = 0.1500$$

$$y(x_4) = y_4$$

$$y(0.20) = 1.0306$$

Put  $n=4$  in (2):

$$y_4 = y_3 + h(2x_3 y_3^2)$$

$$= 1.0306 + 0.05(2(0.20)(1.0306)^2)$$

$$= 1.0518$$

and

$$x_4 = x_0 + 4h$$

$$= 0 + 4(0.05) = 0.2000$$

$$y(0.25) = 1.0518$$

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Put  $n=5$  in (2):

$$y_5 = y_4 + h(2x_4 y_4^2)$$

$$= 1.0518 + 0.05(2(0.25)(1.0518)^2)$$

$$= 1.0795$$

and

$$x_5 = x_0 + 5h$$

$$= 0 + 5(0.05) = 0.25$$

$$\Rightarrow y(0.3) = 1.0795$$

(Answer)

Example:-

Find an approximate value for the solution of initial value problem

$$y' = x + y^2, \quad y(1) = 0.$$

at  $x = 1, 1.1, 1.2, 1.3, 1.4, 1.5$ .

Solution:- The initial value problem is

$$y' = x + y^2, \quad y(1) = 0$$

The Euler's Formula is

$$y_{n+1} = y_n + h f(x_n, y_n) \rightarrow (i)$$

$$\text{As } f(x, y) = x + y^2$$

$$f(x_n, y_n) = x_n + y_n^2$$

 $\Rightarrow$  (i) becomes

$$y_{n+1} = y_n + h(x_n + y_n^2) \rightarrow (2)$$

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In tabular form we can write as

| $n$ | $x_n$ | $y_n$  | $y_{n+1} = y_n + h(x_n + y_n^*)$                  |
|-----|-------|--------|---|
| 0   | 1     | 0      | $y_1 = 0 + 0.1(1 + 0^*) = 0.10$                   |
| 1   | 1.1   | 0.10   | $y_2 = 0.10 + 0.1(1.1 + (0.10)^*)$<br>$= 0.2110$  |
| 2   | 1.2   | 0.2110 | $y_3 = 0.2100 + 0.1(1.2 + (0.211)^*)$<br>$= 0.33$ |
| 3   | 1.3   | 0.33   | $y_4 = 0.33 + 0.1(1.3 + (0.33)^*)$<br>$= 0.4709$  |
| 4   | 1.4   | 0.47   | $y_5 = 0.47 + 0.1(1.4 + (0.47)^*)$<br>$= 0.6321$  |
| 5   | 1.5   |        |   |

$$\Rightarrow y(1.5) = 0.6321$$

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(Answer)

## II HEUN'S METHOD (OR) Improved Euler's Method.

Given  $\frac{dy}{dx} = f(x, y)$  with  $y(x_0) = y_0$

where  $h$  is step-size. Integrate from  $x_n$  to  $x_{n+1}$

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} dx = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

$$y \Big|_{x_n}^{x_{n+1}} = \frac{h}{2} \{ f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \}$$

$\therefore$  using Trapezoidal Rule.

$$y_{n+1} - y_n = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

which is improved Euler's Formula.

But  $f(x_{n+1}, y_{n+1})$  which occurs on R.H.S of equation is unknown. So we first calculate  $y_{n+1}$  from Euler's Method.

which is  $y_{n+1}^* = y_n + h(f(x_n, y_n))$ .

Thus for each stage we have the following two formulas.

Euler's Formula:  $y_{n+1}^* = y_n + h[f(x_n, y_n)]$

Improved Euler's Formula

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]$$

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Example:

Solve  $\frac{dy}{dx} = x^2 + y^2$ ;  $y(0) = 1$ ,  $h = 0.1$

Find  $y(0.1)$  using Euler's Improved Method.

Solution:-  $\frac{dy}{dx} = x^2 + y^2$

$$\Rightarrow f(x, y) = x^2 + y^2$$

$$\Rightarrow f(x_n, y_n) = x_n^2 + y_n^2$$

$$x_0 = 0, y_0 = 1, h = 0.1$$

Euler's Formula is

$$y_{n+1}^* = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1}^* = y_n + h (x_n^2 + y_n^2) \rightarrow (1)$$

Euler's Improved Formula is

$$y_{n+1} = y_n + h [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]$$

$$\Rightarrow y_{n+1} = y_n + h [x_n^2 + y_n^2 + x_{n+1}^2 + y_{n+1}^{*2}] \rightarrow (2)$$

Put  $n = 0$  in (1)

$$y_1^* = y_0 + h (x_0^2 + y_0^2)$$

$$= 1 + 0.1 (0 + 1)$$

$$y_1^* = 1.1$$

and  $x_1 = x_0 + h$

$$= 0 + 0.1 = 0.1$$

Again put  $n = 0$  in (2)

$$y_1 = y_0 + \frac{h}{2} [x_0^2 + y_0^2 + x_1^2 + y_1^{*2}]$$

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$$y_1 = 1 + \frac{(0.1)}{2} [0 + 1 + (0.1)^2 + (1.1)^2]$$

$$y_1 = 1 + 0.0500 (2.2200)$$

$$y_1 = 1.111$$

$$\Rightarrow y(0.1) = 1.111$$

(Ans)

Example:-

$\frac{dy}{dx} = x + y$ ;  $y(0) = 1$ ,  $h = 0.1$

Find  $y(0.2)$  using Improved Euler's Method.

Solution:- Given  $\frac{dy}{dx} = x + y$

$$\therefore f(x, y) = x + y$$

$$f(x_n, y_n) = x_n + y_n$$

$$x_0 = 0, y_0 = 1, h = 0.1$$

Euler's Formula:

$$y_{n+1}^* = y_n + h f(x_n, y_n)$$

$$y_{n+1}^* = y_n + h (x_n + y_n) \rightarrow (i)$$

Improved Euler's Formula:

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]$$

$$y_{n+1} = y_n + \frac{h}{2} (x_n + y_n + x_{n+1} + y_{n+1}^*) \rightarrow (ii)$$

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I<sup>st</sup> iteration Put  $n=0$  in (i) :

$$y_1^* = y_0 + h(x_0 + y_0)$$

$$= 1 + 0.1(0+1) = 1.1$$

$$x_1 = x_0 + h$$

$$x_1 = 0 + 0.1 = 0.1$$

Put  $n=0$  in (ii)

$$y_1 = y_0 + h \left[ x_0 + y_0 + x_1 + y_1^* \right]$$

$$= 1 + \frac{0.1}{2} [0+1+0.1+1.1]$$

$$= 1.11$$

$$\therefore x_1 = 0.1, y_1 = 1.11$$

2<sup>nd</sup> iteration. Put  $n=1$  in (i) :

$$y_2^* = y_1 + h(x_1 + y_1)$$

$$= 1.11 + 0.1(0.1 + 1.11)$$

$$y_2^* = 1.231$$

and

$$x_2 = x_1 + h$$

$$x_2 = 0.1 + 0.1 = 0.2$$

Put  $n=1$  in (ii)

$$y_2 = y_1 + h \left[ x_1 + y_1 + x_2 + y_2^* \right]$$

$$= 1.11 + \frac{0.1}{2} [0.1 + 1.11 + 0.2 + 1.231]$$

$$y_2 = 1.2421$$

$$x_2 = 0.2$$

$$\Rightarrow y(0.2) = 1.2421$$

(Answer)

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### III MODIFIED EULER'S METHOD :-

The modified Euler's formula is

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Example :

Apply Modified Euler's formula to find  $y(0.3)$  given that

$$\frac{dy}{dx} = x + y ; y(0) = 1, h = 0.1$$

Solution :-

$$\frac{dy}{dx} = x + y$$

$$x_0 = 0, y_0 = 1$$

$$f(x, y) = x + y$$

$$f(x_n, y_n) = x_n + y_n$$

Modified Euler's Formula is

$$y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} (x_n + y_n)\right] \quad (*)$$

$$\text{As } f(x, y) = x + y$$

$$f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} (x_n + y_n)\right) = x_n + \frac{h}{2} + y_n + \frac{h}{2} (x_n + y_n)$$

Then \* becomes

$$y_{n+1} = y_n + h \left( x_n + \frac{h}{2} + y_n + \frac{h}{2} (x_n + y_n) \right)$$

$$= y_n + h \left[ \frac{h}{2} [1 + x_n + y_n] + x_n + y_n \right]$$

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$$\Rightarrow y_{n+1} = y_n + h \left[ \frac{h}{2} (1 + x_n + y_n) + x_n + y_n \right] \rightarrow (*)'$$

I<sup>st</sup> Iteration :- Put  $n=0$  in  $(*)'$

$$y_1 = y_0 + h \left[ \frac{h}{2} (1 + x_0 + y_0) + x_0 + y_0 \right]$$

$$= 1 + 0.1 \left[ \frac{0.1}{2} (1 + 0 + 1) + 0 + 1 \right]$$

$$= 1 + 0.11$$

$$= 1.11$$

$$x_1 = x_0 + h$$

$$= 0 + 0.1 = 0.1$$

II<sup>nd</sup> Iteration :- Put  $n=1$  in  $(*)'$

$$y_2 = y_1 + h \left[ \frac{h}{2} (1 + x_1 + y_1) + x_1 + y_1 \right]$$

$$= 1.11 + 0.1 \left[ \frac{0.1}{2} (1 + 1.11 + 0.1) + 0.1 + 1.11 \right]$$

$$= 1.11 + 0.1 (1.3205)$$

$$= 1.2421$$

$$x_2 = x_0 + 2h$$

$$x_2 = 0 + 2(0.1) = 0.2$$

3<sup>rd</sup> Iteration :- Put  $n=2$  in  $(*)'$

$$y_3 = y_2 + h \left[ \frac{h}{2} (1 + x_2 + y_2) + x_2 + y_2 \right]$$

$$= 1.2421 + 0.1 \left[ \frac{0.1}{2} (1 + 0.2 + 1.2421) + 0.2 + 1.2421 \right]$$

$$y_3 = 1.2421 + 0.1 (1.5642)$$

$$y_3 = 1.3985$$

(Answer).

Example :-

$$\text{Solve } \frac{dy}{dx} = x + 2y ; 0 \leq x \leq 0.2$$

$$h = 0.1$$

$$y(0) = 1$$

Using Modified Euler's Method.

Solution :- Give  $\frac{dy}{dx} = x + 2y$

$$f(x, y) = x + 2y$$

$$f(x_n, y_n) = x_n + 2y_n$$

$$x_0 = 0, y_0 = 1$$

we find  $y(0.2)$

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Euler's Modified Formula is

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right) \rightarrow (1)$$

$$f(x_n, y_n) = x_n + 2y_n$$

$$f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} (x_n + 2y_n)\right) = \frac{x_n + \frac{h}{2} + 2\left(y_n + \frac{h}{2} (x_n + 2y_n)\right)}{2}$$

Put in (1) we have,

$$y_{n+1} = y_n + h \left[ \frac{x_n + \frac{h}{2} + 2\left(y_n + \frac{h}{2} (x_n + 2y_n)\right)}{2} \right]$$

$$y_{n+1} = y_n + h \left[ \frac{h}{2} + x_n + 2y_n + h(x_n + 2y_n) \right]$$

$$y_{n+1} = y_n + h \left[ \frac{h}{2} + x_n(1+h) + 2y_n(1+h) \right] \rightarrow (2)$$

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1<sup>st</sup> Iteration :- Put  $n=0$  in (2)

$$y_1 = y_0 + h \left[ \frac{h}{2} + x_0(-1+h) + 2y_0(1+h) \right]$$

$$= 1 + 0.1 \left[ \frac{0.1}{2} + 0(-1+0.1) + 2(1+0.1) \right]$$

$$y_1 = 1 + 0.1 \left[ \frac{0.1}{2} + 2.2 \right]$$

$$y_1 = 1.2250$$

$$\text{And } x_1 = x_0 + h$$

$$= 0 + 0.1 = 0.1$$

$$x_1 = 0.1 \text{ and } y_1 = 1.2250$$

$$y(0.1) = 1.2250$$

2<sup>nd</sup> Iteration :- Put  $n=1$  in (2)

$$y_2 = y_1 + h \left[ \frac{h}{2} + x_1(1+h) + 2y_1(1+h) \right]$$

$$y_2 = 1.2250 + 0.1 \left[ \frac{0.1}{2} + 0.1(1+0.1) + 2(1.2250)(1+0.1) \right]$$

$$= 1.2250 + 0.1 [2.8550]$$

$$= 1.2250 + 0.2855$$

$$y_2 = 1.5105$$

$$x_2 = x_0 + 2h$$

$$= 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 1.5105$$

(Answer).

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## EXERCISE

I Solve following questions by Euler's Method.

i)  $y' = x+y$  ;  $y(0)=0$  ,  $h=0.2$   
Carry out 6 steps.

$$\text{Ans: } 1.18318$$

ii)  $\frac{dy}{dx} = x+y^2$  and  $y=1$  , at  $x=0$ .  
Find  $y(0.5)$   $\therefore h=0.1$

$$\text{Ans: } 1.9346$$

iii) If  $xy' = x+y$  , when  $y$  has at value  $x=1$  is 2.

Find  $y$  when  $x=2$  (Step size 0.1)

iv) Solve  $\frac{dy}{dx} = -xy^2$  ,  $y=2$  at  $x=0$ .

obtain  $y$  at  $x=0.2$  where  $h=0.05$ .

v)  $y' = \frac{-y^2}{1+x^2}$  estimate  $y(0.2)$

with  $y(0)=1$  ,  $h=0.05$ .

vi) Solve  $\frac{dy}{dx} = 1+x \sin(xy)$   $0 \leq x \leq 1$

with  $h=0.1$  and

$$y(0)=0$$



## II Solve following by Improved Euler's Method?

(i) Derive Improved Euler's Method. Apply this to find  $y(0.2)$  from  $y' = -2xy^2$  with  $y(0) = 1$ ,  $h = 0.1$ . Compare your result with exact value.

(ii) Find approximate value of  $y$  when  $x = 0.06$  given  $\frac{dy}{dx} = x^2 + y$ ,  $y(0) = 1$  taking interval  $0.02$ .

Ans: 1.0619

(iii) Tabulate  $y(x)$  for  $x = 0.1, 0.2, 0.3$  from IVP  $y' = 1 + xy$  where  $y(0) = 1$ .

(iv) Find  $y(0.2)$  given

$$\frac{dy}{dx} = x + 3y, \quad y(0) = 1, \quad h = 0.1$$

Ans: 2.492.

## III Solve the following question by Modified Euler's Method?

(i) Use Modified Euler's Method to tabulate  $y(0.3)$  for  $y' = 1 + xy$  where  $y(0) = 1$ .

(ii) Solve  $\frac{dy}{dx} = x^2 + y^2$ ,  $y(0) = 1$ ,  $h = 0.5$  at  $x = 1$  (Hint  $y(1) = ?$ )

(iii) Solve following differential equation  
 $\frac{dy}{dx} = x - y$   $0 \leq x \leq 0.2$   
 $y(0) = 1$

taking  $h = 0.1$  and working to 4 decimal places by modified Euler's Method.  
 (Ans: 0.8372)

Compare with exact value

Hint:  $y(0.2) = ?$

Solution of L.D.E  $y' - y = x$   
 is  $y = x - 1 + 2e^{-x}$

(iv) Show that modified Euler's Method for solution of differential eq.  
 $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$

is given by

$$y_{n+1} = y_n + \frac{h}{2} \{ f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \}$$

Hint: Use Euler's and improved Euler's Method.



## IV TAYLOR'S SERIES

## ALGORITHM

Consider the differential equation  
 $y' = f(x, y)$   
 with I.C  $y(x_0) = y_0$  ]  $\rightarrow$  (i)  
 If  $F(x)$  is exact solution of I.V.P given  
 in (i), then Taylor's series of  
 $F(x)$  around  $x = x_0$  is given by

$$F(x) = F(x_0 + (x - x_0)) \\ = F(x_0) + (x - x_0)F'(x_0) + \frac{(x - x_0)^2}{2!}F''(x_0) \\ + \frac{(x - x_0)^3}{3!}F'''(x_0) + \dots \rightarrow (2)$$

Put  $x = x_1$  and  $y_1 = F(x_1)$

(2)  $\Rightarrow$

$$y_1 = y_0 + (x_1 - x_0)y_0' + \frac{(x_1 - x_0)^2}{2!}y_0'' + \frac{(x_1 - x_0)^3}{3!}y_0''' + \frac{(x_1 - x_0)^4}{4!}y_0^{(iv)} + \dots$$

where  $h = x_1 - x_0$

$$\Rightarrow y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(iv)} + \dots \rightarrow (3)$$

Similarly;

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{(iv)} + \dots$$

so on in general

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{(iv)} + \dots \rightarrow (*)$$

Remark:

(i) Neglecting  $h^2$  and higher power of  $h$  in (\*)

$$y_{n+1} = y_n + hy_n'$$

$y_{n+1} = y_n + hf(x_n, y_n)$  which is Euler's Formula.

(ii) Neglecting  $h^3$  and higher power of  $h$  in (\*)

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' \text{ which is Taylor's Series Method of order 2.}$$

(iii) Neglecting  $h^4$  and higher power of  $h$  in (\*)

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n'''$$

which is Taylor's Series Method of order 3.

(iv) Put  $x_0 = 0$  in (2)

$$F(x) = F(x_0) + (x - x_0)F'(x_0) + \frac{(x - x_0)^2}{2!}F''(x_0) + \dots$$

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \dots$$

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \dots$$

Which is power series in  $x$  of  $y$ .

Example :-

Find value of  $y(0.3)$  for I.V.P  
 $y' = -2xy^2$ ,  $y(0) = 1$ ,  $h = 0.1$   
 using Taylor's series algorithm of order 2.

Solution :-

Taylor's Series algorithm of order 2 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' \rightarrow (1)$$

$$y' = -2xy^2$$

$$y_n' = -2x_n y_n^2$$

$$\begin{aligned} y'' &= -2x(2y y') + (-2)y^2 \\ &= -4xy y' - 2y^2 \\ &= -4xy(-2xy^2) - 2y^2 \\ &= +8x^2 y^3 - 2y^2 \end{aligned}$$

$$y_n'' = -2y_n^2 + 8x_n^2 y_n^3$$

Put in (1) we have

$$y_{n+1} = y_n + h(-2x_n y_n^2) + \frac{h^2}{2!}(-2y_n^2 + 8x_n^2 y_n^3) \rightarrow (2)$$

I<sup>st</sup> Iteration :- Put  $n=0$  in (2)

$$\begin{aligned} y_1 &= y_0 + h(-2x_0 y_0^2) + \frac{h^2}{2!}(-2y_0^2 + 8x_0^2 y_0^3) \\ &= 1 + (0.1)(-2(0)(1)^2) + \frac{(0.1)^2}{2!}(-2(1)^2 + 8(0)(1)^3) \end{aligned}$$

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$$y_1 = 1 + 0 + (-0.0100) + 0$$

$$y_1 = 0.990, \quad x_1 = x_0 + h = 0 + 0.1 = 0.1$$

II<sup>nd</sup> Iteration :- Put  $n=1$  in (2)

$$y_2 = y_1 + h(-2x_1 y_1^2) + \frac{h^2}{2!}(-2y_1^2 + 8x_1^2 y_1^3)$$

$$y_2 = 0.990 + (0.1)(-2(0.1)(0.99)^2) + \frac{(0.1)^2}{2!}[-2(0.99)^2 + 8(0.1)^2(0.99)^3]$$

$$y_2 = 0.961$$

$$x_2 = x_0 + 2h$$

$$= 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 0.961$$

III<sup>rd</sup> Iteration :- Put  $n=2$  in (2)

$$y_3 = y_2 + h(-2x_2 y_2^2) + \frac{h^2}{2!}(-2y_2^2 + 8x_2^2 y_2^3)$$

$$y_3 = 0.961 + (0.1)[-2(0.2)(0.961)^2] + \frac{(0.1)^2}{2!}[-2(0.961)^2 + 8(0.2)^2(0.961)^3]$$

$$y_3 = 0.961244$$

$$x_3 = x_0 + 3h$$

$$= 0 + 3(0.1) = 0.3$$

$$y(0.3) = 0.961244$$

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Exact Value:

$$\frac{dy}{dx} = -2xy^2$$

$$\int \frac{dy}{y^3} = \int -2x dx$$

$$\frac{y^{-1}}{-1} = -\frac{2x^2}{2} + C$$

$$-\frac{1}{y} = x^2 + C$$

Given  $y(0) = 1$ 

$$-\frac{1}{y} = 0 + C$$

$$-1 = C \Rightarrow C = -1$$

$$\Rightarrow -\frac{1}{y} = x^2 - 1$$

$$\Rightarrow \frac{1}{y} = 1 + x^2$$

$$y = \frac{1}{1+x^2}$$

$$y(0.3) = \frac{1}{1+(0.3)^2} = 0.917431$$

$$\text{Error} = 0.917431 - 0.916244$$

$$= 0.001187$$

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Ans.

Example:-

If  $y$  satisfies eq.  $\frac{dy}{dx} = x^2y - 1$ ,  $y(0) = 1$   
 using Taylor's expansion  
 obtain  $y$  as series in power of  $x$ . Also  
 find  $y(0.3)$  to 5 d.p.

Solution:- Taylor's Series

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{(4)} + \dots \rightarrow (1)$$

$$\text{When } x_0 = 0, y_0 = 1$$

$$\text{Given } y' = x^2y - 1, y_0' = x_0^2y_0 - 1$$

$$= 0(1) - 1 = -1$$

$$y'' = 2xy + x^2y', y_0'' = 2x_0y_0' + x_0^2y_0$$

$$= 2(0)(1) + (0)^2(1) = 0$$

$$y''' = 2y + 2xy' + x^2y'' + 2xy''$$

and

$$y_0''' = 2y_0 + 2x_0y_0' + x_0^2y_0'' + 2x_0y_0''$$

$$y_0''' = 2(1) + 2(0)(-1) + (0)^2(0) + 2(0)(0)$$

$$= 2$$

$$y^{(4)} = 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$$

$$= 6y' + 6xy'' + x^2y'''$$

$$y_0^{(4)} = 6y_0' + 6x_0y_0'' + x_0^2y_0'''$$

$$= 6(-1) + 6(0)(0) + (0)^2(2)$$

$$= -6$$

Put all values in (1).

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$$y = 1 + x(-1) + 0 + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$$

$$y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

which is required Taylor's Series expansion of  $y$  in power of  $x$ .

Put  $x = 0.3$

$$y(0.3) = 1 - (0.3) + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \dots$$

$$= 1 - 0.3 + 0.0090 - 0.0020$$

$$= 0.7070$$

Answer.

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\* Example:

Find  $y(0.3)$  for I.V.P  $y' = x + y$   
where  $y(0) = 1$ . Apply Taylor's Series algorithm of order 3 with  $h = 0.1$   
Comparing with exact result.

Solution:- As Taylor's Series Algorithm of order 3 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \rightarrow \text{①}$$

Given  $y' = x + y \Rightarrow y_n' = x_n + y_n$   
 $y'' = 1 + y' \Rightarrow y_n'' = 1 + y_n'$   
 $y_n'' = 1 + x_n + y_n$   
 $y_n''' = 1 + x_n + y_n$

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$$y_n' = x_n + y_n$$

$$y_n'' = 1 + x_n + y_n$$

$$y_n''' = 1 + x_n + y_n$$

Put in ①

$$y_{n+1} = y_n + h(x_n + y_n) + \frac{h^2}{2!}(1 + x_n + y_n) + \frac{h^3}{3!}(1 + x_n + y_n) \rightarrow (*)$$

$$y(0) = 1$$

$$x_0 = 0, y_0 = 1, h = 0.1$$

I<sup>st</sup> Iteration: Put  $n=0$  in (\*)

$$y_1 = y_0 + h(x_0 + y_0) + \frac{h^2}{2!}(1 + x_0 + y_0) + \frac{h^3}{3!}(1 + x_0 + y_0)$$

$$= 1 + (0.1)(0 + 1) + \frac{(0.1)^2}{2!}(1 + 0 + 1) +$$

$$\frac{(0.1)^3}{3!}(1 + 0 + 1)$$

$$y_1 = 1.11033$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\Rightarrow y(0.1) = 1.11033$$

II Iteration: Put  $n=1$  in (\*)

$$y_2 = y_1 + h(x_1 + y_1) + \frac{h^2}{2!}(1 + x_1 + y_1) + \frac{h^3}{3!}(1 + x_1 + y_1)$$

$$= 1.11033 + (0.1)(0.1 + 1.11033) + \frac{(0.1)^2}{2!}(1 + 0.1 + 1.11033)$$

$$+ \frac{(0.1)^3}{3!}(1 + 0.1 + 1.11033)$$

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$$y_2 = 1.2427$$

$$x_2 = x_0 + 2(0.1) = 0.2$$

$$y(0.2) = 1.2427$$

III<sup>rd</sup> Iteration: Put  $n=2$  in (\*)

$$y_3 = y_2 + h(x_2 + y_2) + \frac{h^2}{2!}(1 + x_2 + y_2) + \frac{h^3}{3!}(1 + x_2 + y_2)$$

$$y_3 = 1.2427 + 0.1(0.2 + 1.2427) + \frac{(0.1)^2}{2!}(1 + 0.2 + 1.2427) + \frac{(0.1)^3}{3!}(1 + 0.2 + 1.2427)$$

$$y_3 = 1.3996$$

$$x_3 = x_0 + 3h$$

$$= 0 + 3(0.1)$$

$$= 0.3$$

$$y(0.3) = 1.399679$$

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Ans.

Exact Solution:

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

$$I.F. e^{-\int dx} = e^{-x}$$

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} x$$

$$\int \frac{d}{dx} (e^{-x} y) = \int e^{-x} x$$

$$e^{-x} y = -x e^{-x} - e^{-x} + C$$

$$y = -x + C e^x - 1$$

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using condition

$$y(0) = 1$$

$$1 = -(0) + C e^0 - 1$$

$$2 = C$$

$$\Rightarrow y = -x - 1 + 2e^x$$

$$\text{Put } x = 0.3$$

$$y = -0.3 - 1 + 2e^{(0.3)}$$

$$y = 1.399718$$

$$\text{Error} = \text{Exact} - \text{Approx. Value}$$

$$= 1.39971 - 1.3996$$

$$= 0.000039$$

(Answer)

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# EXERCISE

- (i) Using Taylor's Series algorithm of order 2 find  $y(0.2)$ ,  $y(0.4)$  and  $y(0.6)$  of T.V.P is given that

$$y' = -xy^2, \quad y(0) = 2.$$

Compare with exact value.

- (ii) Find approximate value of  $y(0.1)$ ,  $y(0.2)$  of  $\frac{dy}{dx} = (x+1)y$ ;  $y(0) = 1$ .

using T.S algorithm of order 3 ( $h=0.1$ )

- (iii) Using Taylor's Series find solution of  $y' = -2xy^2$   $y(0) = 1$  at  $x = 0.3$  correct to 5 d.p.

- (iv) Using T.S find solution of  $xy' = x - y$ ;  $y(2) = 2$  at  $x = 2.1$  correct to 5 d.p.  
(Hint:  $y' = 1 - \frac{y}{x}$ )

- (v) Given  $y' = y^2 + 1$ ;  $y(0) = 0$  obtain  $y$  as a series in power of  $x$ . Also find  $y(0.2)$  check your answer with exact solution.

$$\text{Ans: } y(0.2) = 0.202709$$

$$\text{and } y = x + \frac{x^3}{3} + \frac{x^5}{15} + \dots$$

# V RUNGE - KUTTA METHOD (R.K Method)

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$$

Integrate from  $x_n$  to  $x_{n+1}$

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} dx = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

$$y|_{x_n}^{x_{n+1}} = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

by Trapezoidal Rule

$$y(x_{n+1}) - y(x_n) = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_{n+1} - y_n = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$= y_n + \frac{h}{2} [f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))] \rightarrow (i)$$

Using Euler's Formula

$$\text{Let } K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + hf(x_n, y_n))$$

$$\text{or } K_2 = hf(x_n + h, y_n + K_1)$$

put in (i)

$$y_{n+1} = y_n + \frac{1}{2} (K_1 + K_2)$$

which is R.K Formula of order 2.



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Ques:

Runge-Kutta Method of order 2.

$$y_{n+1} = y_n + \frac{1}{2} (K_1 + K_2)$$

$$\text{where } K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + K_1)$$

Ques:

Runge-Kutta Method of order 4

$$y_{n+1} = y_n + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$$

$$K_3 = hf(x_n + \frac{h}{2}, y_n + \frac{K_2}{2})$$

$$K_4 = hf(x_n + h, y_n + K_3)$$

Runge-Kutta Method of order 3.

$$y_{n+1} = y_n + \frac{1}{6} (K_1 + 4K_2 + K_3)$$

where

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$$

$$K_3 = hf(x_n + h, y_n + K_1 + 2K_2)$$

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Example:

Apply Runge formula of order 2 to approximate value of  $y$  when  $x=1.2$  given  $\frac{dy}{dx} = 3x + y^2$  and  $y=1.2$  when  $x=1$

Solution:-  $\frac{dy}{dx} = 3x + y^2$

$$f(x, y) = 3x + y^2$$

$$f(x_n, y_n) = 3x_n + y_n^2$$

R-K method of order 2 is

$$y_{n+1} = y_n + \frac{1}{2} (K_1 + K_2)$$

$$\text{where } K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + K_1)$$

1<sup>st</sup> iteration:-  $x_0 = 1, y_0 = 1.2, h = 0.1$

$$K_1 = hf(x_0, y_0) = h(3x_0 + y_0^2) = 0.1(3(1) + (1.2)^2)$$

$$K_1 = 0.444$$

$$K_2 = hf(x_n + h, y_n + K_1)$$

$$= hf(1 + 0.1, 1.2 + 0.444)$$

$$= hf(1.1, 1.644)$$

$$= 0.1 \{ 3(1.1) + (1.644)^2 \}$$

$$= 0.600$$

Runge's Formula is

$$y_{n+1} = y_n + \frac{1}{2} (K_1 + K_2) \rightarrow (i)$$

Put  $n=0$  in (i)

$$y_1 = y_0 + \frac{1}{2} (0.444 + 0.600)$$

$$y_1 = 1.722$$

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$$x_1 = x_0 + h$$

$$= 1 + 0.1 = 1.1$$

$$\Rightarrow y(1.1) = 1.722$$

2<sup>nd</sup> iteration:-

$$x_1 = 1.1, y_1 = 1.722, h = 0.1$$

$$K_1 = h f(x_1, y_1)$$

$$= h (3x_1 + y_1^2)$$

$$= 0.1 [3(1.1) + (1.722)^2]$$

$$K_1 = 0.6265$$

and

$$K_2 = h f(x_n + h, y_n + K_1)$$

$$= 0.1 f(1.1 + 0.1, 1.722 + 0.6265)$$

$$= 0.1 f(1.2, 2.3485)$$

$$= 0.1 [3(1.2) + (2.3485)^2]$$

$$= 0.1 [3.6 + 5.5155]$$

$$= 0.91155$$

Put  $n = 1$  in (i)

$$y_2 = y_1 + \frac{1}{2} (K_1 + K_2)$$

$$y_2 = 1.722 + \frac{1}{2} (0.6265 + 0.91155)$$

$$= 1.722 + 0.769$$

$$y_2 = 2.491$$

$$x_2 = x_0 + 2h$$

$$= 1 + 2(0.1) = 1.2$$

$$x_2 = 1.2$$

$$y(1.2) = 2.491$$

(Ans) /

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Example:

Solve following differential equation using R.K method of order 4

$$\frac{dy}{dx} = 1 + y^2; y(0) = 0$$

$$h = 0.2$$

Find  $y(0.2)$  and  $y(0.4)$ . Compare with exact value.Solution:- Given  $\frac{dy}{dx} = 1 + y^2$ 

$$f(x, y) = 1 + y^2$$

$$f(x_n, y_n) = 1 + y_n^2$$

$$x_0 = 0, y_0 = 0, h = 0.2$$

Step I: By R.K method of order 4

$$K_1 = h f(x_0, y_0) = h (1 + y_0^2) = 0.2 (1 + 0^2)$$

$$K_1 = 0.2$$

$$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$

$$= (0.2) f(0 + 0.2/2, 0 + 0.2/2)$$

$$= 0.2 f(0.1, 0.1)$$

$$= 0.2 (1 + (0.1)^2)$$

$$= 0.20204$$

$$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$$

$$= h f(0 + \frac{0.2}{2}, 0 + \frac{0.20204}{2})$$

$$= 0.2 f(0.1, 0.101)$$

$$= 0.2 (1 + (0.101)^2)$$

$$= 0.2020402$$

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$$\begin{aligned}
 K_4 &= h f(x_0 + h, y_0 + K_3) \\
 &= 0.2 f(0 + 0.2, 0 + 0.2020402) \\
 &= 0.2 \{ 1 + (0.2020402)^2 \} \\
 &= 0.20816408
 \end{aligned}$$

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) \\
 &= 0 + \frac{1}{6} [0.2 + 2(0.2020) + 2(0.2020402) + 0.20816408] \\
 y_1 &= 0.202707
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= x_0 + h \\
 &= 0 + 0.2 = 0.2 \\
 \Rightarrow y(0.2) &= 0.202707 \quad (\text{Ans}).
 \end{aligned}$$

Step II :-

$$\begin{aligned}
 x_1 &= 0.2, y_1 = 0.202707, h = 0.2 \\
 K_1 &= h f(x_1, y_1) \\
 &= 0.2 f(0.2, 0.202707) \\
 &= 0.2 \{ 1 + (0.202707)^2 \} \\
 &= 0.2082181
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= h f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) \\
 &= h f(0.2 + \frac{0.2}{2}, 0.202707 + \frac{0.2082181}{2}) \\
 &= h f(0.3, 0.306816) \\
 &= 0.2 \{ 1 + (0.306816)^2 \} \\
 &= 0.2182272.
 \end{aligned}$$

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$$\begin{aligned}
 K_3 &= h f(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) \\
 &= 0.2 f(0.2 + \frac{0.2}{2}, 0.202707 + \frac{0.2182272}{2}) \\
 &= 0.2 f(0.3, 0.3121206) \\
 &= 0.2 \{ 1 + (0.3121206)^2 \} \\
 &= 0.2194839.
 \end{aligned}$$

$$\begin{aligned}
 K_4 &= h f(x_1 + h, y_1 + K_3) \\
 &= 0.2 f(0.2 + 0.2, 0.202707 + 0.2194839) \\
 &= 0.2 f(0.4, 0.4221909) \\
 &= 0.2 [1 + (0.4221909)^2] \\
 &= 0.235649.
 \end{aligned}$$

$$\begin{aligned}
 y_2 &= y_1 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4] \\
 &= 0.202707 + \frac{1}{6} [0.2082181 + 2(0.2182272) + 2(0.2194839) + 0.235649] \\
 y_2 &= 0.422791
 \end{aligned}$$

$$\begin{aligned}
 x_2 &= x_0 + 2h \\
 &= 0 + 2(0.2) = 0.4 \\
 y(0.4) &= 0.422791
 \end{aligned}$$

Exact Value :

$$\frac{dy}{dx} = 1 + y^2$$

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{dy}{1+y^2} = \int dx$$

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$$\tan^{-1}(y) = x + c$$

$$y(0) = 0$$

$$\tan^{-1}(0) = 0 + c$$

$$\Rightarrow c = 0$$

$$\Rightarrow \tan^{-1} y = x$$

$$y = \tan x$$

$$y(0.2) = \tan(0.2)$$

$$= 0.2027$$

$$\text{and } y(0.4) = \tan(0.4)$$

$$= 0.4227932$$

Example :-

(Answer)

Show that most popular R.K method of order 4 reduces to Simpson's Rule in  $\frac{dy}{dx}$  is a function of  $x$  alone.

$$\text{Solution :- } \frac{dy}{dx} = f(x)$$

$$1. K_1 = hf(x_n) = hf_n$$

$$K_2 = hf(x_n + \frac{h}{2}) = hf_{n+\frac{h}{2}}$$

$$K_3 = hf(x_n + h) = hf_{n+h}$$

$$K_4 = hf(x_n + h) = hf_{n+h}$$

Thus

$$y_{n+1} = y_n + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

$$= y_n + \frac{1}{6} [hf_n + 2hf_{n+\frac{h}{2}} + 2hf_{n+\frac{h}{2}} + hf_{n+h}]$$

$$y_{n+1} = y_n + \frac{h}{6} (f_n + 4f_{n+\frac{h}{2}} + f_{n+h})$$

which is Simpson's Rule with step size  $\frac{h}{2}$ .  
Replace  $h$  by  $2h$ .

$$y_{n+1} = y_n + \frac{h}{3} (f_n + 4f_{n+\frac{h}{2}} + f_{n+h})$$

$$y_{n+1} = y_n + \frac{h}{3} \{f(x_n) + 4f(x_n + h) + f(x_n + 2h)\}$$

which is Simpson's  $\frac{1}{3}$  Rule.

(proved)

Example :-

Use Runge-Kutta 4<sup>th</sup> order method to find an approximate value of  $y$  for  $x=0.2$  in step of 0.1 if  $\frac{dy}{dx} = x+y^2$  given that  $y(0)=1$ .

$$\text{Solution :- } \frac{dy}{dx} = x+y^2, \quad y(0)=1$$

$$f(x, y) = x+y^2$$

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

Step I:

$$K_1 = hf(x_0, y_0) = h(x_0 + y_0^2) \\ = 0.1(0 + 1^2) \\ = 0.100$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) \\ = hf(0 + 0.05, 0.100 + 0.05) \\ = hf(0.05, 1.05)$$

$$= 0.1(0.05 + (1.05)^2) \\ = 0.1152$$

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$$k_3 = hf(x_0 + h, y_0 + k_2)$$

$$= 0.1 f\left(\frac{0+0.1}{2}, \frac{0.1152+1}{2}\right)$$

$$= 0.1 f(0.05, 1.0576)$$

$$= 0.1 [0.05 + (1.0576)^2]$$

$$= 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1 f(0+0.1, 1+0.1168)$$

$$= 0.1 [0.1 + (1.1168)^2]$$

$$= 0.1347$$

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6} [0.100 + 2(0.1152) + 2(0.1168) + 0.1347]$$

$$= 1.1165$$

$$x_1 = x_0 + h$$

$$= 0 + 0.1 = 0.1$$

$$y(0.1) = 1.1165$$

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Step II:  $x_1 = 0.1, y_1 = 1.1165, h = 0.1$

$$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1165)$$

$$= 0.1 [0.1 + (1.1165)^2]$$

$$= 0.1347$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left[\frac{0.1+0.1}{2}, \frac{1.1165+0.1347}{2}\right]$$

$$= 0.1 [f(0.15, 1.1838)]$$

$$= 0.1 [0.15 + (1.1838)^2]$$

$$= 0.1551$$

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$$k_3 = hf(x_1 + h, y_1 + k_2)$$

$$= 0.1 f\left(\frac{0.1+0.1}{2}, \frac{1.1165+0.1551}{2}\right)$$

$$= 0.1 f(0.15, 1.194)$$

$$= 0.1 [0.15 + (1.194)^2]$$

$$= 0.1576$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1 f(0.1+0.1, 1.1165+0.1576)$$

$$= 0.1 f(0.2, 1.2741)$$

$$= 0.1 [0.2 + (1.2741)^2]$$

$$= 0.1823$$

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1165 + \frac{1}{6} [0.1347 + 2(0.1551) + 2(0.1576) + (0.1823)]$$

$$= 1.2736$$

$$x_2 = x_0 + 2h$$

$$= 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 1.2736$$

(Answer).

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## EXERCISE

- (i) Derive R-K method of order 2 and use it to find  $y(0.02)$  and  $y(0.03)$  with  $h=0.01$  of problem  
 $y' = 2y + x$ ,  $y(0) = 1$ .
- (ii) Apply Runge's Formula (2<sup>nd</sup> order) to find an approximate value of  $y$  when  $x = 1.1$  given that  
 $\frac{dy}{dx} = x - y$  &  $y = 1$  when  $x = 1$ .  
 Ans: 1.005
- (iii) Calculate  $y(0.1)$  and  $y(0.2)$  from I.V.P  
 $y' = x + 2y$ ,  $y(0) = 0.75$   
 taking  $h = 0.1$   
 Comparing with exact value.  
 Ans:  $y(0.1) = 0.9$   
 $y(0.2) = 1.1384$
- (iv) Solve equation  $y' = x + y$ ,  $y(0) = 0$  at  $x = 0.5$  Taking  $h = 0.1$  compare your result with exact value
- (v) Apply Runge's formula (2<sup>nd</sup> order) to find value of  $y$  when  $x = 0.02$  given that  $\frac{dy}{dx} = x^2 + y$  &  $y(0) = 1$   
 Ans: 1.0202

(vi) Solve I.V.P by Runge's Formula of order 4 of following.

- (i) Obtain  $y$  when  $x = 1.1$  given that  $y = 1.2$  when  $x = 1$  and  $y$  satisfies equation

$$\frac{dy}{dx} = 3x + y^2$$

Ans: 1.7271

- (ii) Find  $y(0.2)$  for eq.  $\frac{dy}{dx} = y - x$ ,  $y(0) = 1$   
 Take  $h = 0.2$

Ans: 1.167845

- (iii) At  $x = 0.8$  find approx. value of  $y$  given that  $y = 0.41$  when  $x = 0.4$  and  
 $\frac{dy}{dx} = \sqrt{x+y}$

Ans: 0.8489

- (iv) Find  $y(0.2)$  given

$$\frac{dy}{dx} = 3x + \frac{1}{2}y, \quad y(0) = 1, \quad h = 0.1$$

Ans: 1.17492

- (v) Use R.K Method find  $y$  when  $x = 1.2$  in step size 0.1 given that

$$(i) \frac{dy}{dx} = \frac{y}{x} \text{ and } y(1) = 1$$

Ans: 1.2

$$(ii) \frac{dy}{dx} = x^2 + y^2, \quad y(1) = 1.5$$

Ans: 2

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## PREDICTOR-CORRECTOR METHODS

The methods discussed so far, to solve differential equations, numerically were self-starting one step method. To apply these methods, we were required information only at the beginning of interval. But, now in present section we shall discuss Predictor-corrector methods which require function value at point  $x_0, x_{n-1}, x_{n-2}, \dots$  for the computation of function at  $x_{n+1}$ . And these function value can be obtained by using Euler's method, Taylor's method, R.K method of order 4 etc.

A predictor formula is used to predict the value of  $y$  at  $x_{n+1}$  and then a corrector formula is used to improve the value of  $y$ . Here we discuss two methods as following.

- (i) Milne's Method
- (ii) Adam-Bashforth Method.

## (i) MILNE'S METHOD :-

to solve IVP

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y(x_0) = y_0$$

$h = \text{Step size}$

by Milne's method, we first find the approximate value of  $y_{n+1}$  by predictor formula and then improve it by corrector formula.

In terms of  $f$ , Milne's Predictor and corrector formula are

Predictor formula :

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3} [2f_{n-2} - f_{n-1} + 2f_n] \quad \rightarrow (A)$$

Corrector formula :

$$y_{n+1} = y_{n-1} + \frac{h}{3} [f_{n-1} + 4f_n + \tilde{f}_{n+1}]$$

where  $h$  is step size.

and  $\tilde{f}_{n+1} = f(x_{n+1}, \tilde{y}_{n+1})$

(Also  $\tilde{y}_{n+1} = y'_{n+1}$ )

$P$  stands for predictor.

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Example:

Find  $y(0.4)$  and  $y(0.5)$  by Milne's method from  $\frac{dy}{dx} = xy$ ;  $y(0)=1$   
 $h=0.1$

Solution :- The given IVP is  $\frac{dy}{dx} = xy$ ;  $y(0)=1$   
 $h=0.1$

By R.K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \rightarrow (i)$$

Where  $k_1 = hf(x_n, y_n)$   
 $k_2 = hf(x_n + \frac{h}{2}, y_n + k_1)$   
 $k_3 = hf(x_n + \frac{h}{2}, y_n + k_1)$   
 $k_4 = hf(x_n + h, y_n + k_3)$

For  $n=0$ ,

$$k_1 = hf(x_0, y_0) = h(x_0 y_0) = (0.1)[(0)(1)] = 0$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$= h(x_0 + \frac{h}{2})(y_0 + \frac{k_1}{2})$$

$$= (0.1)(0+0.5)(1+0) = 0.005$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$= h(x_0 + \frac{h}{2})(y_0 + \frac{k_2}{2})$$

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$$= (0.1)(0+0.5)(1+0.0025) = 0.0050125$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h(x_0 + h)(y_0 + k_3)$$

$$= (0.1)(0+0.1)(1+0.0050125)$$

$$= 0.005013$$

So by (i) we have

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6} [0 + 2(0.005) + 2(0.0050125) + 0.005013]$$

$$y(0.1) = 1.00501252$$

For  $n=1$ :-

$$k_1 = hf(x_1, y_1)$$

$$= h x_1 y_1 = (0.1)(0.1)(1.005013)$$

$$= 0.010050$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$= h(x_1 + \frac{h}{2})(y_1 + \frac{k_1}{2})$$

$$= (0.1)(0.1+0.5)(1.005013 + \frac{0.01005}{2})$$

$$= (0.1)(0.15)(1.010038) = 0.015151$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$= h(x_1 + \frac{h}{2})(y_1 + \frac{k_2}{2})$$

$$= (0.1)(0.1+0.5)(1.005013 + \frac{0.015151}{2})$$

$$= 0.015189$$

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$$\begin{aligned}
 k_4 &= hf(x_1+h, y_1+k_3) \\
 &= h(x_1+h)(y_1+k_3) \\
 &= (0.1)(0.1+0.1)(1.005013+0.015189) \\
 &= 0.020404
 \end{aligned}$$

So (i) gives

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.005013 + \frac{1}{6} [0.0065 + 2(0.01515 + 0.015189) + 0.020404]$$

$$= 1.005013 + \frac{1}{6} (0.099134)$$

$$= 1.005013 + 0.015189$$

$$y(0.2) = 1.020202$$

For  $n=2$  :-

$$k_1 = hf(x_2, y_2)$$

$$= h(x_2)(y_2) = (0.1)(0.2)(1.020202)$$

$$= 0.020404$$

$$k_2 = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2})$$

$$= (0.1)(0.2+0.5)(1.020202 + \frac{0.020404}{2})$$

$$= 0.025760$$

$$k_3 = hf(x_2 + h, y_2 + k_2)$$

$$= h(x_2 + \frac{h}{2})(y_2 + \frac{k_2}{2})$$

$$= (0.1)(0.2+0.5)(1.020202 + \frac{0.025760}{2})$$

$$= 0.025827$$

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$$\begin{aligned}
 k_4 &= hf(x_2+h, y_2+k_3) \\
 &= h(x_2+h)(y_2+k_3) \\
 &= (0.1)(0.1+0.1)(1.020202+0.025827) \\
 &= 0.03138
 \end{aligned}$$

So (i) gives

$$y_3 = y_2 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$= 1.020202 + \frac{1}{6} [0.020404 + 2(0.02576 + 0.025827) + 0.031381]$$

$$= 1.020202 + \frac{1}{6} (0.154959)$$

$$= 1.020202 + 0.025827$$

$$y(0.3) = 1.046029$$

For  $n=3$  :-

$$k_1 = hf(x_3, y_3) = h(x_3)(y_3)$$

$$= (0.1)(0.3)(1.046029)$$

$$= 0.031381$$

$$k_2 = hf(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2})$$

$$= h(x_3 + \frac{h}{2})(y_3 + \frac{k_1}{2})$$

$$= (0.1)(0.3+0.5)(1.046029 + \frac{0.031381}{2})$$

$$= 0.037160$$

$$k_3 = hf(x_3 + h, y_3 + \frac{k_2}{2})$$

$$= h(x_3 + \frac{h}{2})(y_3 + \frac{k_2}{2})$$

$$= (0.1)(0.3+0.5)(1.046029 + \frac{0.037160}{2})$$

$$= 0.037261$$

$$k_4 = hf(x_3 + h, y_3 + k_3)$$

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$$= h(x_3 + h)(y_3 + k_3)$$

$$= (0.1)(0.3 + 0.1)(1.046029 + 0.037261)$$

$$= 0.043332$$

So (i) gives

$$y_4 = y_3 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$= 1.046029 + \frac{1}{6} [0.03138 + 2(0.037160 + 0.037261) + 0.043332]$$

$$= 1.046029 + \frac{1}{6} (0.223555)$$

$$y(0.4) = 1.083288$$

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For  $n=4$ :

$$k_1 = hf(x_4, y_4) = h(x_4)(y_4)$$

$$= (0.1)(0.4)(1.083288)$$

$$= 0.043332$$

$$k_2 = hf(x_4 + \frac{h}{2}, y_4 + \frac{k_1}{2})$$

$$= h(x_4 + \frac{h}{2})(y_4 + \frac{k_1}{2})$$

$$= (0.1)(0.4 + 0.05)(1.083288 + 0.043332)$$

$$= 0.049723$$

$$k_3 = hf(x_4 + \frac{h}{2}, y_4 + \frac{k_2}{2})$$

$$= (0.1)(0.4 + 0.05)(1.083288 + 0.049723)$$

$$= 0.049867$$

$$k_4 = hf(x_4 + h, y_4 + k_3)$$

$$= h(x_4 + h)(y_4 + k_3)$$

$$= (0.1)(0.4 + 0.1)(1.083288 + 0.049867)$$

$$= 0.056658$$

So (i) gives

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$$y_5 = y_4 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$= 1.083288 + \frac{1}{6} [0.043332 + 2(0.049723 + 0.049867) + 0.056658]$$

$$= 1.083288 + \frac{1}{6} (0.299170)$$

$$y(0.5) = 1.133150$$

Now

$$f(x_n, y_n) = x_n y_n$$

$$f(x_1, y_1) = x_1 y_1 = (0.1)(1.005013)$$

$$= 0.100501$$

$$f(x_2, y_2) = x_2 y_2 = (0.2)(1.020202)$$

$$= 0.20404$$

$$f_3 = f(x_3, y_3) = x_3 y_3 = (0.3)(1.046029)$$

$$= 0.313809$$

$$f_4 = x_4 y_4 = (0.4)(1.083288)$$

$$= 0.433315$$

$$f_5 = x_5 y_5 = (0.5)(1.133150)$$

$$= 0.566575$$

So by Milne's Predictor and  
Corrector formula, we have

For  $n=4$ : By Predictor formula, (A)

$$\tilde{y}_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4(0.1)}{3} [2(0.100501) - 0.20404 + 2(0.313809)]$$

$$\tilde{y}_4 = 1.083277$$

$$\tilde{f}_4 = x_4 \tilde{y}_4 = (0.4)(1.083277)$$

$$= 0.433311$$

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And by corrector formula by (B)

$$y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + \tilde{f}_4]$$

$$= 1.020202 + \frac{0.1}{3} [0.20404 + 4(0.313809)$$

$$+ 0.433311]$$

$$= 1.020202 + 0.1 (1.892587)$$

$$y(0.4) = 1.083288$$

For  $n=4$ :

By Predictor formula, (A)

$$\tilde{y}_5 = y_1 + 4h [2f_2 - f_3 + 2f_4]$$

$$= 1.005013 + \frac{4(0.1)}{3} [2(0.20404) - 0.313809$$

$$+ 2(0.433315)]$$

$$= 1.005013 + \frac{0.4}{3} (0.960901)$$

$$= 1.133133$$

$$\Rightarrow \tilde{f}_5 = x_5 \tilde{y}_5$$

$$= (0.5)(1.133133)$$

$$= 0.566567$$

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By corrector formula, (B)

$$y_5 = y_3 + \frac{h}{3} [f_3 + 4f_4 + \tilde{f}_5]$$

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$$= 1.046029 + \frac{0.1}{3} [0.313809 + 4(0.433315)$$

$$+ 0.566567]$$

$$= 1.046029 + \frac{0.1}{3} [2.613636]$$

$$y(0.5) = 1.13315$$

(Answer)

Example :-

Find  $y(0.4)$  by Milne's method  
from  $\frac{dy}{dx} = x+y$ ,  $y(0)=1$   
 $h=0.1$ 

Solution:-

The given IVP is

$$\frac{dy}{dx} = x+y ; y(0)=1$$

$$h=0.1$$

Then by Taylor's Method of order 4 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{(4)} \rightarrow 0$$

For  $n=0$ :

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)}$$

$$\text{Given } y' = x+y \Rightarrow y_0' = x_0 + y_0 = 0+1 = 1$$

$$y'' = 1+y' \Rightarrow y_0'' = 1+y_0'$$

$$= 1+1$$

$$= 2$$

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And by corrector formula by (B)

$$y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + \tilde{f}_4]$$

$$= 1.020202 + \frac{0.1}{3} [0.20404 + 4(0.313809) + 0.433311]$$

$$= 1.020202 + \frac{0.1}{3} (1.892587)$$

$$y(0.4) = 1.083288$$

For  $n=4$ :

By Predictor formula, (A)

$$\tilde{y}_5 = y_1 + 4h [2f_2 - f_3 + 2f_4]$$

$$= 1.005013 + \frac{4(0.1)}{3} [2(0.20404) - 0.313809 + 2(0.433315)]$$

$$= 1.005013 + \frac{0.4}{3} (0.960901)$$

$$= 1.133133$$

$$\Rightarrow \tilde{f}_5 = x_5 \tilde{y}_5$$

$$= (0.5)(1.133133)$$

$$= 0.566567$$

By corrector formula, (B)

$$y_5 = y_3 + \frac{h}{3} [f_3 + 4f_4 + \tilde{f}_5]$$

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$$= 1.046029 + \frac{0.1}{3} [0.313809 + 4(0.433315) + 0.566567]$$

$$= 1.046029 + \frac{0.1}{3} [2.613636]$$

$$y(0.5) = 1.13315$$

(Answer)

Example :-

Find  $y(0.4)$  by Milne's method from  $\frac{dy}{dx} = x+y$ ,  $y(0) = 1$ ,  $h = 0.1$ 

Solution:-

The given IVP is

$$\frac{dy}{dx} = x+y; \quad y(0) = 1, \quad h = 0.1$$

Then by Taylor's Method of order 4 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{(4)} \rightarrow 0$$

For  $n=0$ :

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)}$$

$$\text{Given } y' = x+y \Rightarrow y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y'' = 1+y' \Rightarrow y_0'' = 1+y_0' = 1+1 = 2$$

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$$y'' = y'' = 2$$

$$y'' = y_0'' = 2$$

$$y'' = y_0'' = y_0'' = 2$$

So by (i) gives

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2(2)}{2!} + \frac{(0.1)^3(2)}{3!} + \frac{(0.1)^4(2)}{4!}$$

$$= 1 + 0.1 + 0.01 + 0.000333 + 0.000008$$

$$= 1.110341$$

$$y(0.1) = 1.110341$$

For  $n=1$ :

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{(iv)} \rightarrow (ii)$$

$$y_1' = x_1 + y_1$$

$$= 0.1 + 1.110341 = 1.210341$$

$$y_1'' = 1 + y_1'$$

$$= 1 + 1.210341 = 2.210341$$

$$y_1''' = y_1'' = 2.210341$$

$$y_1^{(iv)} = y_1''' = 2.210341$$

So (ii) gives

$$y_2 = 1.110341 + (0.1)(1.210341) + \frac{(0.1)^2(2.210341)}{2!}$$

$$+ \frac{(0.1)^3(2.210341)}{3!} + \frac{(0.1)^4(2.210341)}{4!}$$

$$y(0.2) = 1.242804$$

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For  $n=2$ :

$$y_3 = y_2 + h y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{(iv)} \rightarrow (iii)$$

$$y_2' = x_2 + y_2 = 0.2 + 1.242804$$

$$= 1.442804$$

$$y_2'' = 1 + y_2' = 1 + 1.442804 = 2.442804$$

$$y_2''' = y_2'' = 2.442804$$

$$y_2^{(iv)} = y_2''' = 2.442804$$

So (iii) gives

$$y_3 = 1.242804 + (0.1)(1.442804) + \frac{(0.1)^2(2.442804)}{2!}$$

$$+ \frac{(0.1)^3(2.442804)}{3!} + \frac{(0.1)^4(2.442804)}{4!}$$

$$= 1.242804 + 0.144280 + 0.012214 +$$

$$0.000407 + 0.000010$$

$$y(0.3) = 1.399715$$

For  $n=4$ :

$$y_4 = y_3 + h y_3' + \frac{h^2}{2!} y_3'' + \frac{h^3}{3!} y_3''' + \frac{h^4}{4!} y_3^{(iv)} \rightarrow (iv)$$

$$y_3' = x_3 + y_3 = 0.3 + 1.399715 = 1.699715$$

$$y_3'' = 1 + y_3' = 1 + 1.699715 = 2.699715$$

$$y_3''' = y_3'' = 2.699715$$

$$y_3^{(iv)} = y_3''' = 2.699715$$

So (iv) gives



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$$y_4 = 1.399715 + (0.1)(1.699715) + \frac{(0.1)^2}{2!}(2.397715) + \frac{(0.1)^3}{3!}(2.397715) + \frac{(0.1)^4}{4!}(2.397715)$$

$$y(0.4) = 1.399715 + 0.169972 + 0.011999 + 0.000400 + 0.000010$$

$$y(0.4) = 1.582096$$

Now by Milne's Predictor and Corrector formula we have

For  $n=3$ :

By Predictor formula

$$\tilde{y}_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

Now

$$f(x_1, y_1) = f_1 = x_1 + y_1 = 0.1 + 1.110341 = 1.210341$$

$$f_2 = x_2 + y_2 = 0.2 + 1.242804 = 1.442804$$

$$f_3 = x_3 + y_3 = 0.3 + 1.399715 = 1.699715$$

So

$$\tilde{y}_4 = 1 + \frac{4(0.1)}{3} [2(1.210341) - 1.442804 + 2(1.699715)]$$

$$= 1 + \frac{0.4}{3} [4.377308]$$

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$$\tilde{y}_4 = 1.583641$$

$$\text{Now } \tilde{f}_4 = x_4 + \tilde{y}_4 = 0.4 + 1.583641 = 1.983641$$

By Corrector formula

$$y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + \tilde{f}_4]$$

$$= 1.242804 + \frac{0.1}{3} [1.442804 + 4(1.699715) + 1.983641]$$

$$= 1.242804 + \frac{0.1}{3} (10.225305)$$

$$\Rightarrow y(0.4) = 1.583648$$

(Answer)

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(ii) ADAM-BASHFORTH METHOD :-

Predictor formula :

$$y_{n+1}^p = y_n + h \left[ \frac{55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}}{24} \right] \rightarrow (i)$$

Corrector formula :

$$y_{n+1} = y_n + h \left[ \frac{9f_{n+1}^p + 19f_n - 5f_{n-1} + f_{n-2}}{24} \right] \rightarrow (ii)$$

Where  $f_{n+1}^p = f(x_{n+1}, y_{n+1}^p)$ .\* Example:Solve  $xy' = x - y$  to find  $y(2.2)$   
and  $y(2.25)$ given  $y(2) = 2$ ,  $y(2.05) = 2.00061$ ,  
 $y(2.1) = 2.002385$ ,  $y(2.15) = 2.005242$ 

Solution:  $xy' = x - y$   
 $y' = \frac{x - y}{x}$

$$\Rightarrow y' = 1 - \frac{y}{x}$$

$$f(x, y) = 1 - \frac{y}{x}$$

$$\begin{array}{lll} x_0 = 2 & x_1 = 2.05 & x_2 = 2.1 \\ y_0 = 2 & y_1 = 2.00061 & y_2 = 2.002385 \end{array}$$

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$$x_3 = 2.15$$

$$y_3 = 2.005242$$

$$f_0 = f(x_0, y_0) = f(2, 2) = 1 - \frac{2}{2} = 1 - 1 = 0$$

$$\begin{aligned} f_1 &= f(x_1, y_1) = f(2.05, 2.00061) \\ &= 1 - \frac{2.00061}{2.05} = 0.0240927 \end{aligned}$$

$$\begin{aligned} f_2 &= f(x_2, y_2) = f(2.1, 2.002385) \\ &= 1 - \frac{2.002385}{2.1} \end{aligned}$$

$$= 0.0464834$$

$$\begin{aligned} f_3 &= f(x_3, y_3) = f(2.15, 2.005242) \\ &= 1 - \frac{2.005242}{2.15} \end{aligned}$$

$$= 0.0673294$$

Adam Bashforth Predictor formula

$$y_{n+1}^p = y_n + h \left[ \frac{55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}}{24} \right] \rightarrow (i)$$

and Corrector formula is

$$y_{n+1} = y_n + h \left[ \frac{9f_{n+1}^p + 19f_n - 5f_{n-1} + f_{n-2}}{24} \right] \rightarrow (ii)$$

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Put  $n=3$  in (i)

$$y_4^p = y_3 + \frac{h}{24} [55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$y_4^p = 2.005242 + \frac{0.05}{24} [55(0.0673294) - 59(0.0464834) + 37(0.0240927) - 9(0)]$$

$$\Rightarrow y_4^p = 2.0091003$$

$$\begin{aligned} f_4^p &= f(x_4, y_4^p) = f(2.2, 2.0091003) \\ &= 1 - \frac{2.009603}{2.2} \\ &= 0.0867726 \end{aligned}$$

Again put  $n=3$  in Corrector formula (ii)

$$y_4 = y_3 + \frac{h}{24} [9f_4^p + 19f_3 - 5f_2 + f_1]$$

$$y_4 = 2.00524 + \frac{0.05}{24} [9(0.0867726) + 19(0.0673294) - 5(0.0464834) + 0.0240927]$$

$$y_4 = 2.005242 + 0.003858$$

$$y_4 = 2.0091$$

$$\Rightarrow y(2.2) = 2.0091 \quad (\text{Answer})$$

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Put  $n=4$  in Predictor formula (i)

$$y_5^p = y_4 + \frac{h}{24} [55f_4 - 59f_3 + 37f_2 - 9f_1]$$

$$y_5^p = 2.00913 + \frac{0.05}{24} [55(0.0867728) - 59(0.0673294) + 37(0.0464834) - 9(0.0240927)]$$

$$y_5^p = 2.01392$$

$$\begin{aligned} f_5^p &= f(x_5, y_5^p) \\ &= f(2.25, 2.01392) \\ &= 1 - \frac{2.01392}{2.25} \\ &= 0.1049245 \end{aligned}$$

Again put  $n=4$  in Corrector formula (ii)

$$y_5 = y_4 + \frac{h}{24} [9f_5^p + 19f_4 - 5f_3 + f_2]$$

$$y_5 = 2.0091 + \frac{0.05}{24} [9(0.1049245) + 19(0.0867728) - 5(0.0673294) + 0.0464834]$$

$$y_5 = 2.01389$$

$$\Rightarrow y(2.25) = 2.01389$$

(Answer)

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"Exercise"

Apply Milne's and Adam-Bashforth predictor corrector formulae on following questions.

ii) Solve  $y' = -xy^2$  at  $x = 0.8$  and  $x = 1.0$

using starting values

$$y(0.2) = 1.92308$$

$$y(0.4) = 1.72414, \quad y(0.6) = 1.47059$$

$$\text{Ans: } y(0.8) = 1.21808,$$

$$y(1.0) = 1.0$$

(ii)  $2 \frac{dy}{dx} = (1+x^2)y^2$  and  $y(0) = 1$

$$y(0.1) = 1.06$$

$$y(0.2) = 1.12$$

$$y(0.3) = 1.21$$

Evaluate  $y(0.4)$ .

$$\text{Ans: } 1.2797$$

(iii) Solve  $y' = y^2$  where  $y(1) = 1$  and  $h = 0.1$

find value of IVP at  $x = 1.5$

$$\text{Ans: } y(1.5) = 2$$

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System of differential equations.

A set of simultaneous equations that involves two or more unknown function and their derivatives is called system of differential equations.

For example:

$$\frac{dx}{dt} = 2x + y + 2$$

$$\frac{dy}{dt} = x - y + 2$$

Here  $t$  is independent variable and  $x, y$  are dependent variable.

Runge-Kutta Method of order 4 :-

To solve equations

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

we find  $k_1, k_2, k_3, k_4$  and  $l_1, l_2, l_3, l_4$  in the order



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$$k_1 = hf(t_n, x_n, y_n)$$

$$l_1 = hg(t_n, x_n, y_n)$$

$$k_2 = hf(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2})$$

$$l_2 = hg(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2})$$

$$k_3 = hf(t_n + h, x_n + k_2, y_n + l_2)$$

$$l_3 = hg(t_n + h, x_n + k_2, y_n + l_2)$$

$$k_4 = hf(t_n + h, x_n + k_3, y_n + l_3)$$

$$l_4 = hg(t_n + h, x_n + k_3, y_n + l_3)$$

Finally we obtain

$$x_{n+1} = x_n + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_{n+1} = y_n + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

Runge-kutta Method of order 2:-

$$k_1 = hf(t_n, x_n, y_n)$$

$$l_1 = hg(t_n, x_n, y_n)$$

$$k_2 = hf(t_n + h, x_n + k_1, y_n + l_1)$$

$$l_2 = hg(t_n + h, x_n + k_1, y_n + l_1)$$

$$x_{n+1} = x_n + \frac{1}{2} (k_1 + k_2)$$

$$y_{n+1} = y_n + \frac{1}{2} (l_1 + l_2)$$

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Example :-

Solve initial value problem at  $t=1.2$  where

$$\frac{dy}{dt} = x - 2y, \quad \frac{dx}{dt} = 2x + y.$$

$$x(1) = 2, \quad y(1) = 3.$$

Solution:-  $f(t, x, y) = 2x + y$   
 $g(t, x, y) = x - 2y$

$$\therefore f(t_n, x_n, y_n) = 2x_n + y_n$$

$$g(t_n, x_n, y_n) = x_n - 2y_n$$

1st Iteration:-

$$t_0 = 1, \quad x_0 = 2, \quad y_0 = 3$$

Let  $h = 0.1$

$$k_1 = hf(t_0, x_0, y_0)$$

$$= 0.1 (2x_0 + y_0)$$

$$= 0.1 \{ 2(2) + 3 \}$$

$$= 0.7$$

$$l_1 = hg(t_0, x_0, y_0)$$

$$= h (x_0 - 2y_0)$$

$$= 0.1 \{ 2 - 2(3) \}$$

$$= -0.4$$

$$k_2 = hf(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2})$$

$$= h \{ 2(x_0 + \frac{k_1}{2}) + y_0 + \frac{l_1}{2} \}$$

$$= (0.1) \{ 2 \times 2.35 + 2.8 \} = 0.75$$

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$$l_2 = h g(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2})$$

$$= 0.1 g(1.05, 2.35, 2.8)$$

$$= 0.1 \{ 2.35 - 2(2.8) \}$$

$$= -0.325$$

$$k_3 = h f(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2})$$

$$= 0.1 f(1.05, 2.375, 2.8375)$$

$$= 0.1 \{ 2 \times 2.375 + 2.8375 \}$$

$$= 0.75875$$

$$l_3 = h g(t_0 + h, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2})$$

$$= 0.1 g(1.05, 2.375, 2.8375)$$

$$= 0.1 \{ 2.375 - 2(2.8375) \}$$

$$= -0.33$$

$$k_4 = h f(t_0 + h, x_0 + k_3, y_0 + l_3)$$

$$= 0.1 f(1.1, 2.75875, 2.67)$$

$$= 0.1 \{ 2(2.7587) + 2.67 \}$$

$$= 0.81875$$

$$l_4 = h g(t_0 + h, x_0 + k_3, y_0 + l_3)$$

$$= 0.1 g(1.1, 2.75875, 2.67)$$

$$= 0.1 \{ 2.75875 - 2(2.67) \}$$

$$= -0.25812$$

$$x_1 = x_0 + \frac{1}{6} (k_1 + k_2 + 2k_3 + k_4)$$

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$$x_1 = 2 + \frac{1}{6} \{ 0.7 + 2(0.75 + 0.75875) + 0.81875 \}$$

$$= 2.75604$$

$$y_1 = y_0 + \frac{1}{6} [l_1 + 2(l_2 + l_3) + l_4]$$

$$= 3 + \frac{1}{6} [-0.4 + 2(-0.325 - 0.33) - 0.258125]$$

$$= 2.67198$$

2<sup>nd</sup> Iteration:-

$$t_1 = t_0 + h = 1 + 0.1 = 1.1$$

$$x_1 = 2.75604$$

$$y_1 = 2.67198$$

$$k_1 = h f(t_1, x_1, y_1)$$

$$l_1 = h g(t_1, x_1, y_1)$$

So on Same Solve as in 1<sup>st</sup> iteration

replace  $x_0, y_0$  by  $x_1, y_1$  and  
find  $k_1, k_2, k_3, k_4$  and  
 $l_1, l_2, l_3, l_4$

we get

$$x_2 = 3.650467$$

$$y_2 = 2.47814$$

(Answer)



\* Example:

Solve the equations

$$\frac{dy}{dz} = xz + 1, \quad \frac{dz}{dx} = -xy$$

for  $x = 0.3, 0.3, 0.9$ Given  $y=0, z=1$ , when  $x=0$ .Solution:  $x = 0.3, 0.3, 0.9$  means  
 $h = 0.3$ 1st find value at 0.3 then at 0.6  
and 0.9.Note that  $x$  is independent and  $y, z$   
are dependent variables.

$$f(x, y, z) = xz + 1, \quad g(x, y, z) = -xy$$

$$f(x_0, y_0, z_0) = x_0 z_0 + 1, \quad g(x_0, y_0, z_0) = -x_0 y_0$$

1st iteration:-

$$x_0 = 0$$

$$y_0 = 0$$

$$z_0 = 1$$

$$h = 0.3$$

$$k_1 = h f(x_0, y_0, z_0)$$

$$= h [x_0 z_0 + 1]$$

$$= h [(0)(1) + 1]$$

$$= 0.3 [0 + 1] = 0.3$$

$$l_1 = h g(x_0, y_0, z_0)$$

$$= h [-x_0 y_0]$$

$$= 0.3 [-(0)(0)]$$

$$= 0$$

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$$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$$

$$= 0.3 f(0.15, 0.15, 1)$$

$$= 0.3 \{ 0.15 \times 1 + 1 \} = 0.3450$$

$$l_2 = h g(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$$

$$= 0.3 g(0.15, 0.15, 1)$$

$$= 0.3 [-0.15 \times 0.15]$$

$$= -0.00675$$

$$k_3 = h f(x_0 + h, y_0 + k_2, z_0 + \frac{l_2}{2})$$

$$= 0.3 f(0.15, 0.1725, 0.9966)$$

$$= 0.3 [0.15 \times 0.9966 + 1]$$

$$= 0.3448$$

$$l_3 = h g(x_0 + h, y_0 + k_2, z_0 + \frac{l_2}{2})$$

$$= 0.3 g(0.15, 0.1725, 0.9966)$$

$$= 0.3 [-0.15 \times 0.1725]$$

$$= -0.00776$$

$$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3 f(0.3, 0.3448, 0.99224)$$

$$= 0.3 [0.3 \times 0.99224 + 1]$$

$$= 0.3893$$

$$l_4 = h g(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3 g(0.3, 0.3448, 0.99224)$$

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$$= 0.3 \{-0.3 \times 0.3448\}$$

$$= -0.031036$$

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0 + \frac{1}{6} [0.3 + 2(0.3450 + 0.3448) + 0.3893]$$

$$= 0.3448$$

$$z_1 = z_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

$$= \frac{1}{6} + \frac{1}{6} [0 + 2(-0.00675 + 0.3893) + (-0.031036)]$$

$$= 1 - 0.001001$$

$$= 0.9899$$

$$\Rightarrow y(0.3) = 0.3448$$

II<sup>nd</sup> Iteration :-

$$x_1 = x_0 + h = 0 + 0.3 = 0.3$$

$$y_1 = 0.3448, z_1 = 0.9899$$

By using R-k method order 4 formula we have

$$k_1 = 0.3891, l_1 = -0.031042$$

$$k_2 = 0.43155, l_2 = -0.07281$$

$$k_3 = 0.42873, l_3 = 0.07568$$

$$k_4 = 0.4646, l_4 = -0.1393$$

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$$y_2 = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.3448 + \frac{1}{6} (0.3891 + 2(0.43155 + 0.4287) + 0.4646)$$

$$= 0.3448 + 0.4290$$

$$y_2 = 0.7738$$

$$z_2 = z_1 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0.9899 + (-0.0788)$$

$$= 0.9121$$

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3<sup>rd</sup> Iteration :-

$$x_2 = x_1 + h = 0.3 + 0.3 = 0.6$$

$$y_2 = 0.7738$$

$$z_2 = 0.921$$

$$k_1 = 0.4642, l_1 = -0.1393$$

$$k_2 = 0.4896, l_2 = -0.2263$$

$$k_3 = 0.47976, l_3 = -0.2292$$

$$k_4 = 0.4844, l_4 = -0.3386$$

$$y_3 = y_2 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.7738 + 0.4812 = 1.2550$$

$$z_3 = z_2 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0.9121 - 0.23148 = 0.6806$$

$$\Rightarrow y(0.9) = 1.2550 \quad (\text{Answer})$$



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Exercise

Solve the following system of equations  
Find  $x$  and  $y$  two steps after the  
given value of  $t$ .

$$(a) \quad \frac{dx}{dt} = x + y - t$$

$$\frac{dy}{dt} = 3x - y + 2t \quad x(0) = 1, y(0) = 2$$

Ans:  $x_1 = 1.31722, y_1 = 2.15000$   
 $x_2 = 1.67828, y_2 = 2.40134$   
 where  $h = 0.1$

$$(b) \quad \frac{dx}{dt} = x^2 + y, \quad \frac{dy}{dt} = 2x - 3y + t$$

$$x(0) = 0.2, y(0) = 0.1$$

Ans:  $x_1 = 0.21495, y_1 = 0.11450$   
 $h = 0.1$

$$(c) \quad \frac{dx}{dt} = 0.1x + 0.2y - 0.3t$$

$$\frac{dy}{dt} = 0.2x - 0.1y + 0.2t$$

at  $t = 1.1$  where  $x(1) = 1, y(1) = 2$ .

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